

# CEPAO shakedown, new computational aspects

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# Introduction

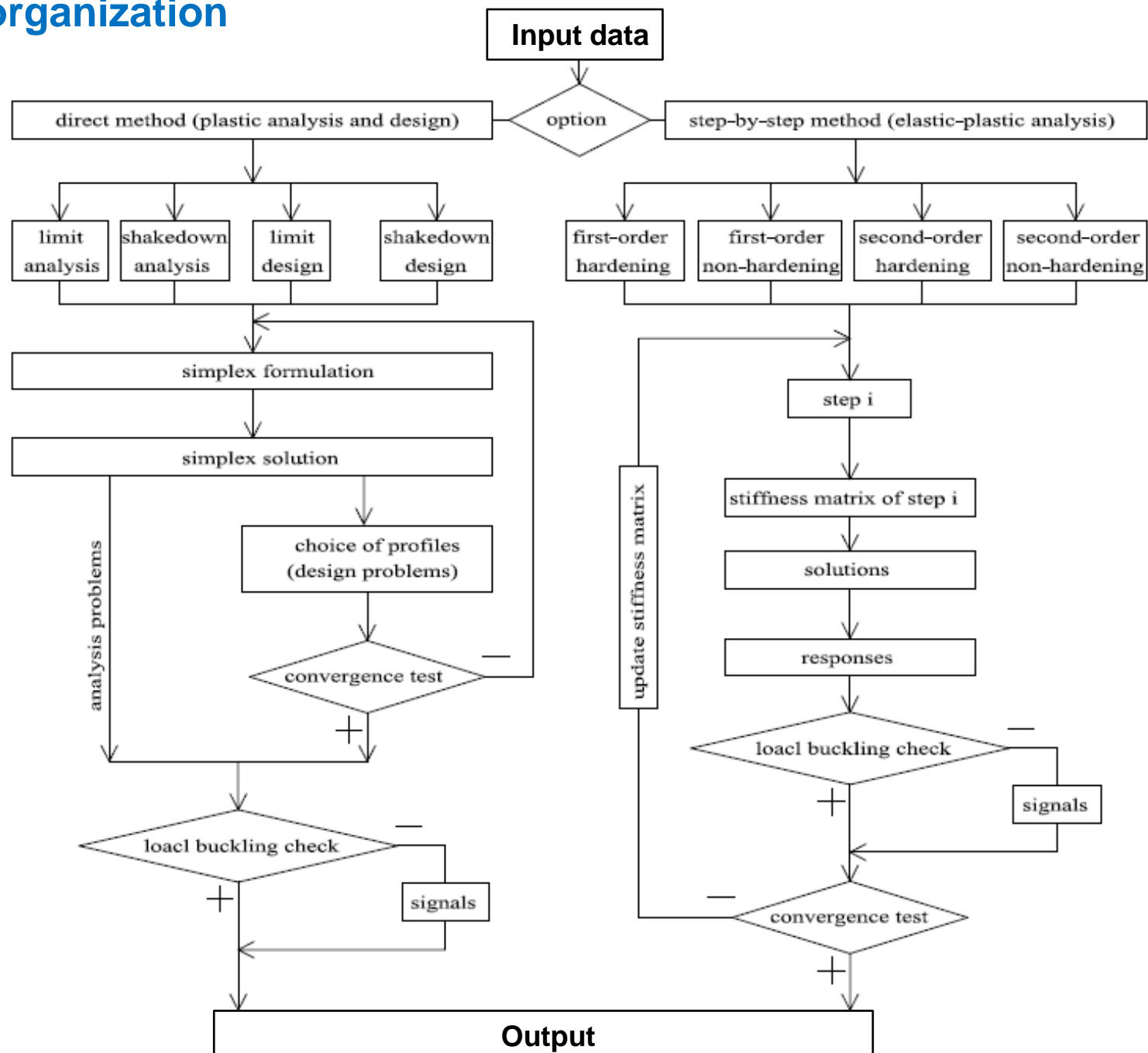
**CEPAO** package

(**C**alcul **E**lasto-**P**lastique, **A**nalyse-**O**ptimisation)

**Built-up in 1980's at University of Liège (Belgium)  
for 2D framed structures**

**New development: 3D framed structures**

# Global organization



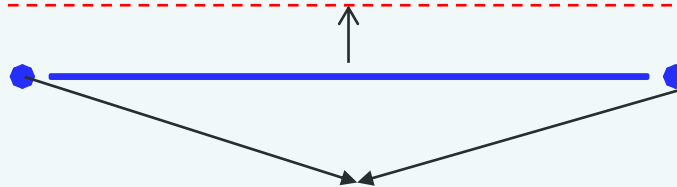
# General features

- The one/two/three-linear behaviors of the mild steel are considered.
- The frames are submitted to fixed or repeated load.
- The second-order effect is taken into account.
- The beam-to-column joints could be rigid or semi-rigid.
- The compact or slender cross-sections are examined.
- The investigation is carried out using direct or step-by-step methods.
- Both analysis and optimization methodologies are applied.

# Element features

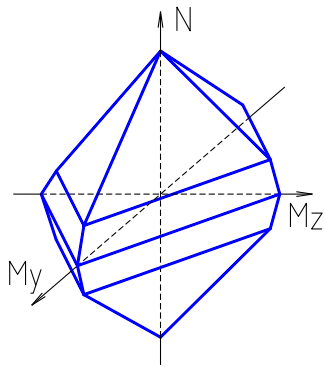
- Formulation by using the elastic Bernoulli beam theory
- P- $\delta$  effect by using stability functions (in step-by-step analysis)
- Imperfection effect by using European buckling curves (in step-by-step analysis)

3D beam-column element



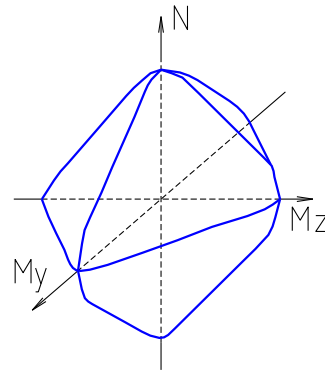
3D plastic hinges

Yield surface



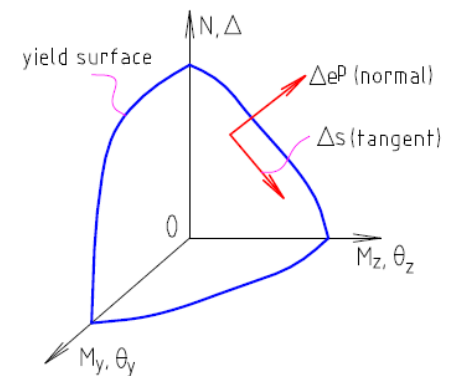
**16 facet-polyherdron**  
(direct method)

or



**Orbison -1982**  
(Single, smooth, convex:  
step-by-step method)

Constitutive law



**Normality rule:**  $\mathbf{e}^p = \lambda \mathbf{N}_C$

# Rigid-plastic formulations

*Limit analysis by kinematical approach*

$$\text{Min } \phi = \mathbf{s}_0^T \boldsymbol{\lambda} \quad \left| \quad \begin{array}{l} \mathbf{N}_C \boldsymbol{\lambda} - \mathbf{B} \mathbf{d} = \mathbf{0} \\ \mathbf{f}^T \mathbf{d} = \xi \\ \boldsymbol{\lambda} \geq \mathbf{0} \end{array} \right.$$

*Shakedown analysis by kinematical approach*

$$\text{Min } Z(\bar{\mathbf{n}}_p, \boldsymbol{\rho}) \equiv \mathbf{n}_p^T \bar{\mathbf{l}} \quad \left| \quad \begin{array}{l} \mathbf{B}^T \boldsymbol{\rho} = \mathbf{0} \\ \mathbf{N}_C^T (\mathbf{s}_e + \boldsymbol{\rho}) \leq \mathbf{n}_p \end{array} \right.$$

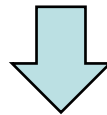
*Limit design by statical approach*

$$\text{Min } Z(\bar{\mathbf{n}}_p, \mathbf{s}) \quad \left| \quad \begin{array}{l} \mathbf{B}^T \mathbf{s} = \mathbf{f} \\ \mathbf{N}_c^T \mathbf{s} \leq \bar{\mathbf{n}}_p \end{array} \right.$$

*Shakedown design by statical approach*

$$\text{Min } \phi = \mathbf{s}_0^T \boldsymbol{\lambda} \quad \left| \quad \begin{array}{l} \mathbf{N} \boldsymbol{\lambda} - \mathbf{B} \mathbf{d} = \mathbf{0} \\ \mathbf{s}_E^T \mathbf{N}_C \boldsymbol{\lambda} = \xi \\ \boldsymbol{\lambda} \geq \mathbf{0} \end{array} \right.$$

Linear programming formulation



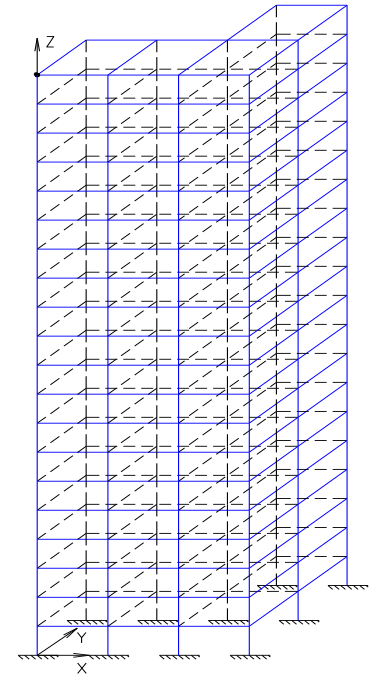
$$\text{Min } \pi = \mathbf{c}^T \mathbf{x} \quad \left| \quad \mathbf{W} \mathbf{x} = \mathbf{b} \right.$$

# Rigid-plastic formulations

$$\text{Min } \pi = \mathbf{c}^T \mathbf{x} \mid \mathbf{W}\mathbf{x} = \mathbf{b}$$

Very large dimension

20 story frame	
Problem	W dimension
Analysis	(921)x(18 961)
Optimization	(16 380)x(19 220)



Several techniques have been proposed and adopted in order to reduce the problem sizes

# Elastic-plastic formulations

Elastic constitutive relation

$$\rightarrow \Delta \mathbf{s} = \mathbf{D} \Delta \mathbf{e}$$

**Elastic**  
**Plastic**

$$\begin{bmatrix} \Delta \mathbf{s}_R \\ \Delta \mathbf{s}_C \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{RR} & \mathbf{D}_{RC} \\ \mathbf{D}_{RC}^T & \mathbf{D}_{CC} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{e}_R - 0 \\ \Delta \mathbf{e}_C - \Delta \mathbf{e}_C^p \end{bmatrix}$$

Normality rule

$$\Delta \mathbf{e}_C^p = \mathbf{N}_C \Delta \lambda$$

Plastic constitutive

$$\mathbf{N}_C^T \Delta \mathbf{s}_C - H F \mathbf{N}_C' \Delta \lambda = 0$$

Elastic-plastic constitutive relation

$$\begin{bmatrix} \Delta \mathbf{s}_R \\ \Delta \mathbf{s}_C \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{RR} - \mathbf{D}_{RC} \mathbf{N}_C \mathbf{R}_1 & \mathbf{D}_{RC} - \mathbf{D}_{RC} \mathbf{N}_C \mathbf{R}_2 \\ \mathbf{D}_{RC}^T - \mathbf{D}_{CC} \mathbf{N}_C \mathbf{R}_1 & \mathbf{D}_{CC} - \mathbf{D}_{CC} \mathbf{N}_C \mathbf{R}_2 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{e}_R \\ \Delta \mathbf{e}_C \end{bmatrix}$$

$$\mathbf{K} = \mathbf{B}^T \boxed{\mathbf{D}} \mathbf{B}$$



# Numerical examples

## I. 3-D rigid frames analysis

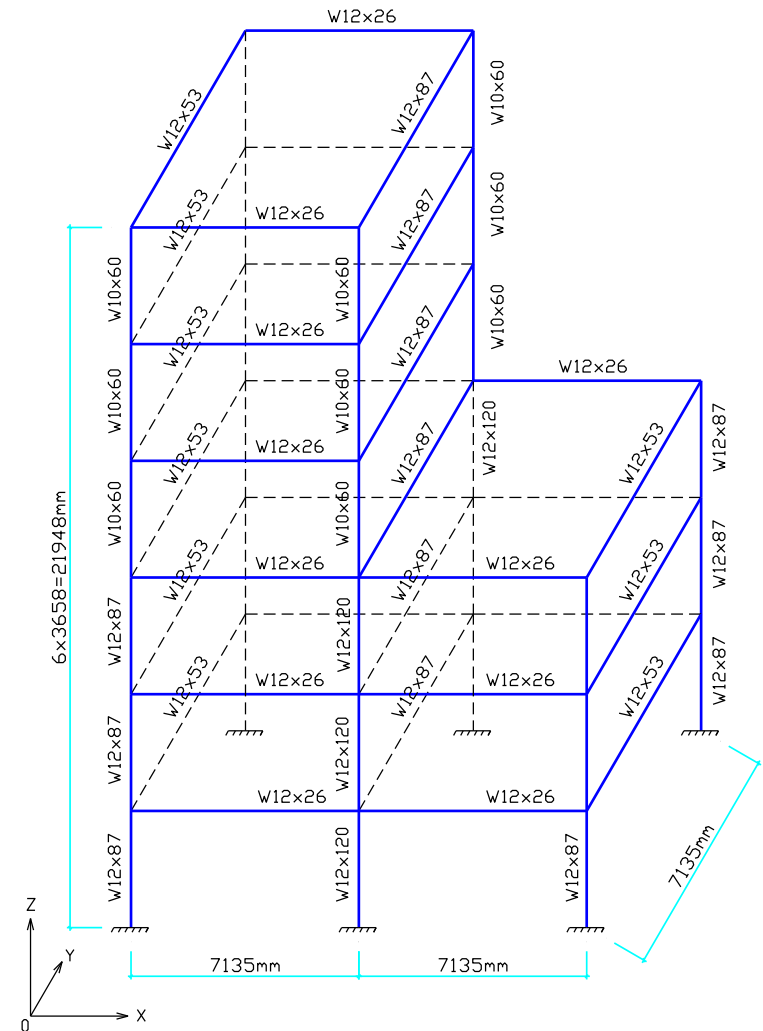
### Frame I.1 – Six-story space frame

$\sigma_p = 250 \text{ Mpa}$

$E = 206000 \text{ Mpa}$

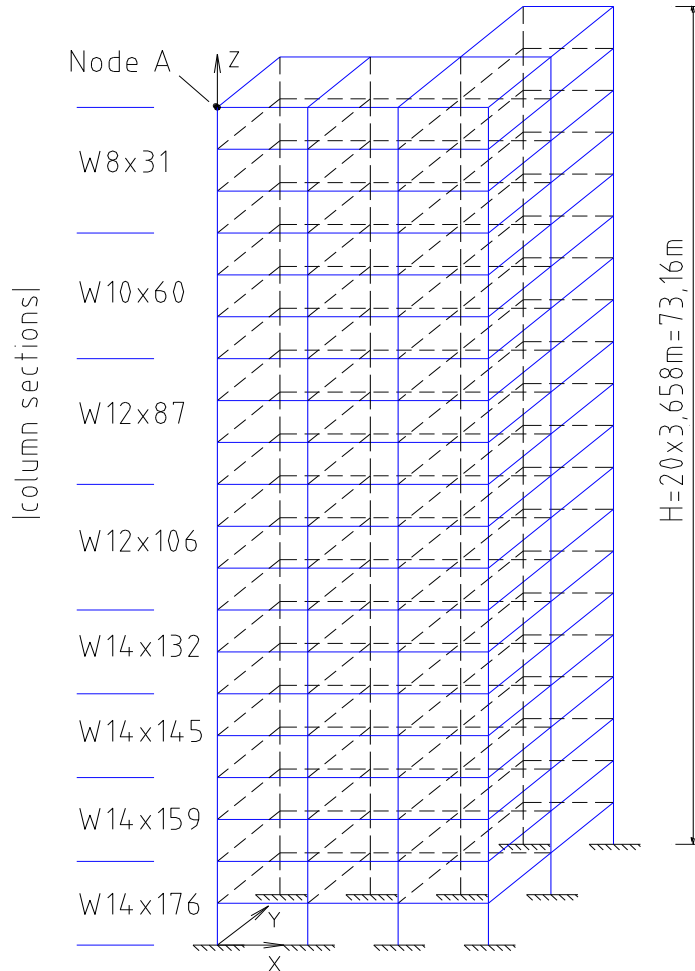
Floor pressure:  $4.8 \text{ kN/m}^2$

Wind load (Y direction):  $26.7 \text{ kN/node}$



# Numerical examples

## Frame I.2 – Twenty-story space frame

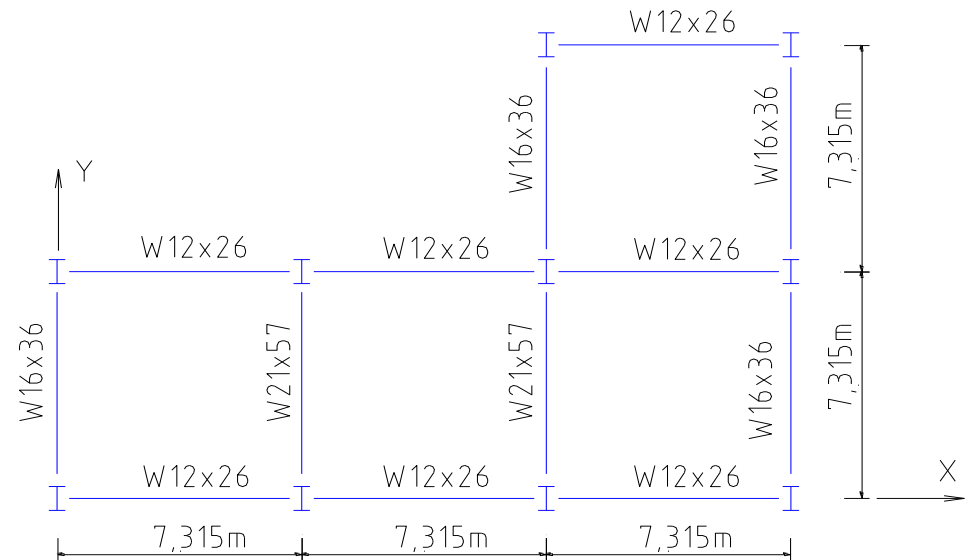


$\sigma_p = 345 \text{ Mpa}$

$E = 200000 \text{ Mpa}$

Floor pressure:  $4.8 \text{ kN/m}^2$

Wind load (Y direction):  $0.96 \text{ kN/m}^2$



# Numerical examples

Load multipliers given by second-order analysis

Author	Model	Frame I.1	Frame I.2
Liew JYR - 2000	Plastic-hinge	2.010	-
Kim SE - 2001	Plastic-hinge	2.066	-
Cuong NH - 2006	Fiber-plastic-hinge	2.066	1.003
Liew JYR - 2001	Plastic-hinge	-	1.031
Jiang XM - 2002	Fibre-element	-	1.000
Chorean C.G.- 2005	Distributed plasticity, n=300	1.998	1.005
	n=30	2.124	1.062
CEPAO	Plastic-hinge, hardening ignored	2.033	1.024
	hardening considered	2.149	1.051

# Numerical examples

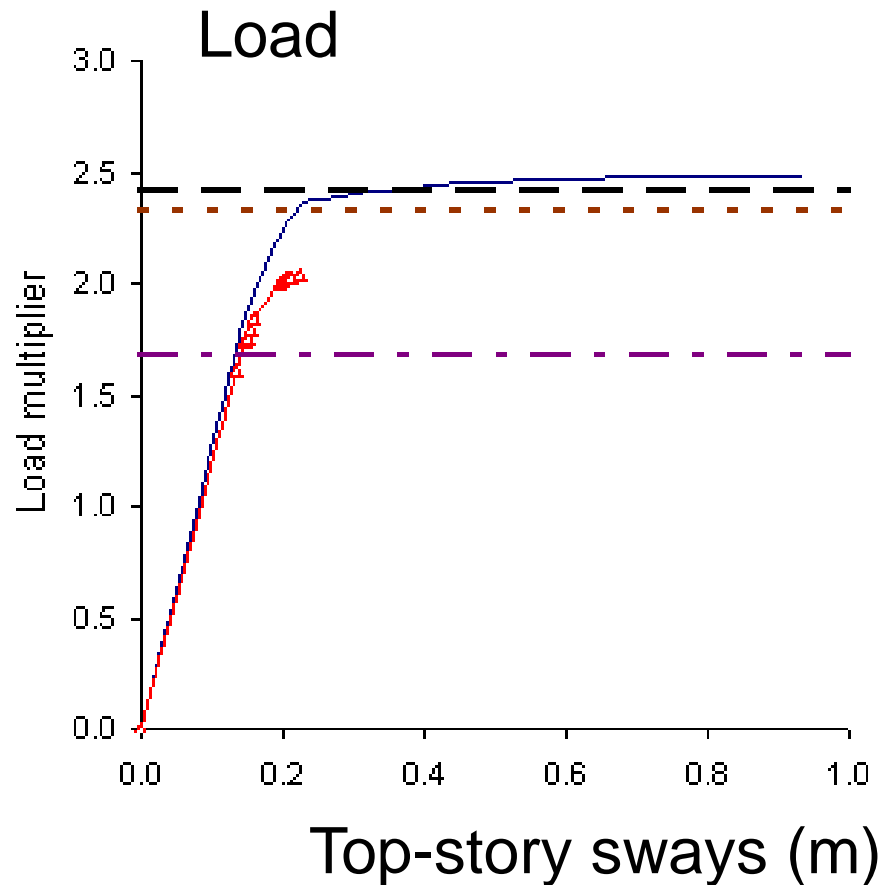
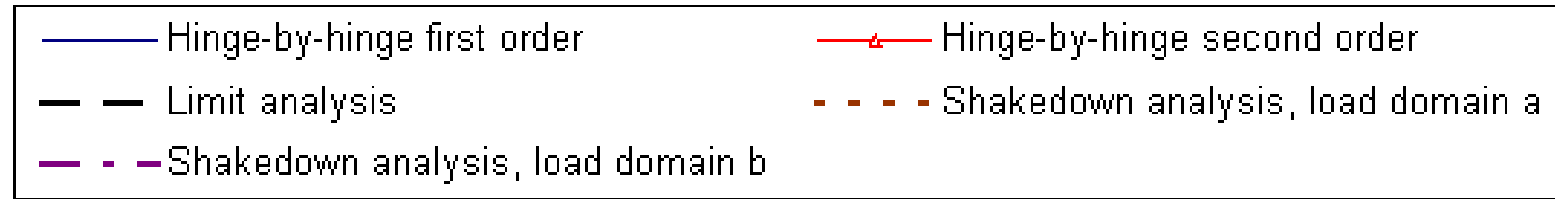
## Load multipliers given CEPAO

Method, model	Frame 1	Frame 2	
Elastic-plastic, first-order	2.489	1.689	(instantaneous)
Elastic-plastic, second-order	2.033	1.024	(unstableness)
Limit analysis	2.412	1.698	(instantaneous)
Shakedown analysis, load a	2.311	1.614	(incremental)
Shakedown analysis, load b	1.670	0.987	(Alternating)

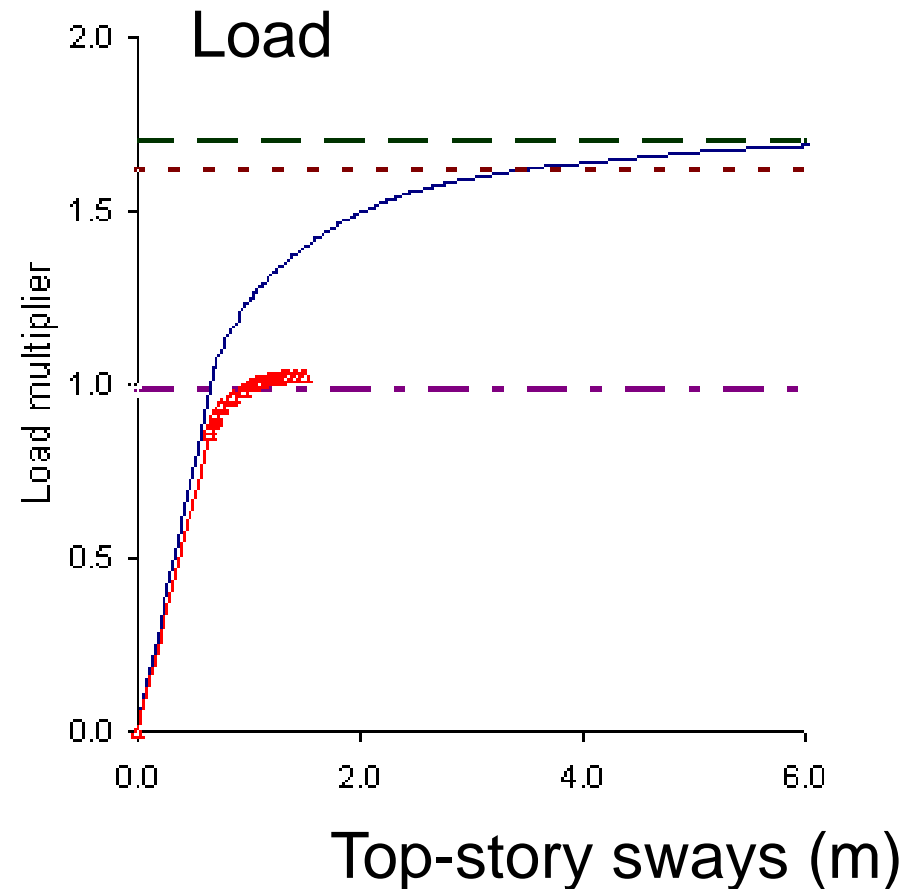
load a:  $0 \leq \text{floor pressure} \leq 4.8 \beta \text{ (kN/m}^2\text{)}; 0 \leq \text{wind load} \leq 0.96 \beta \text{ (kN/m}^2\text{)}$

load b:  $0 \leq \text{floor pressure} \leq 4.8 \beta \text{ (kN/m}^2\text{)}; -0.96\beta \leq \text{wind load} \leq 0.96\beta \text{ (kN/m}^2\text{)}$

# Numerical examples

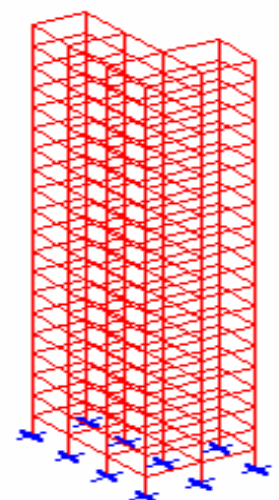
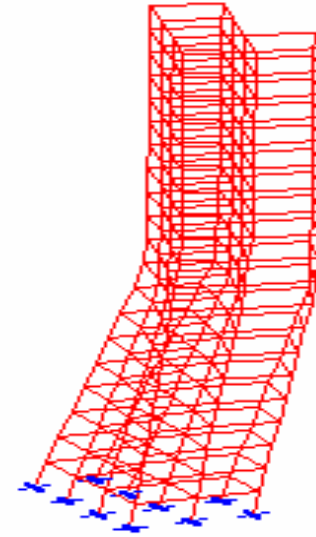
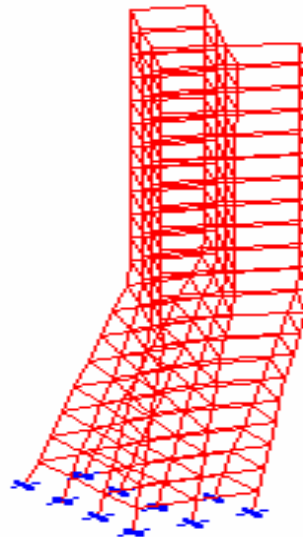
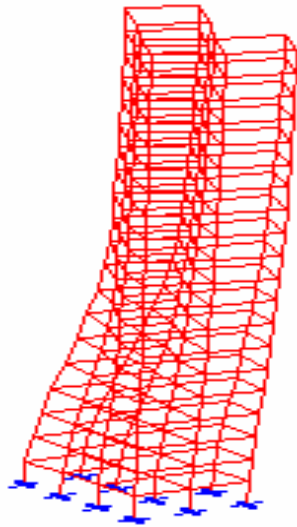
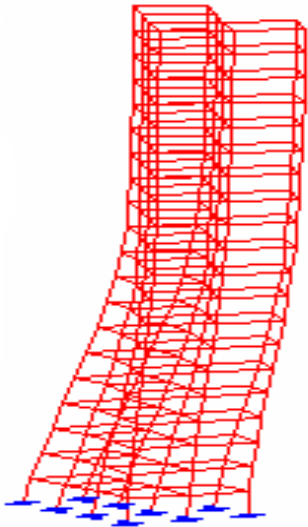
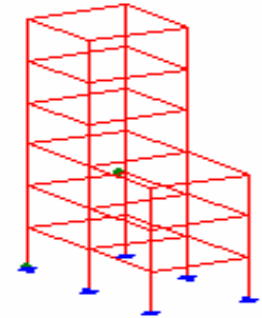
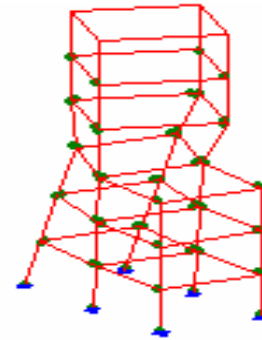
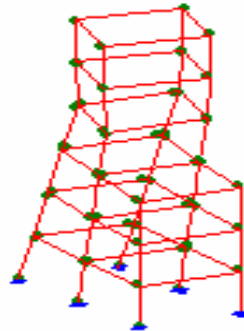
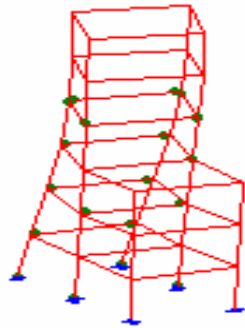
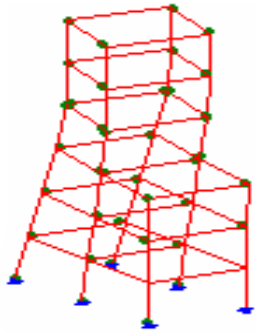


**Frame I.1**



**Frame I.2**

# Numerical examples



Instantaneous  
mechanism  
(first-order)

unstablenss  
(second-order)

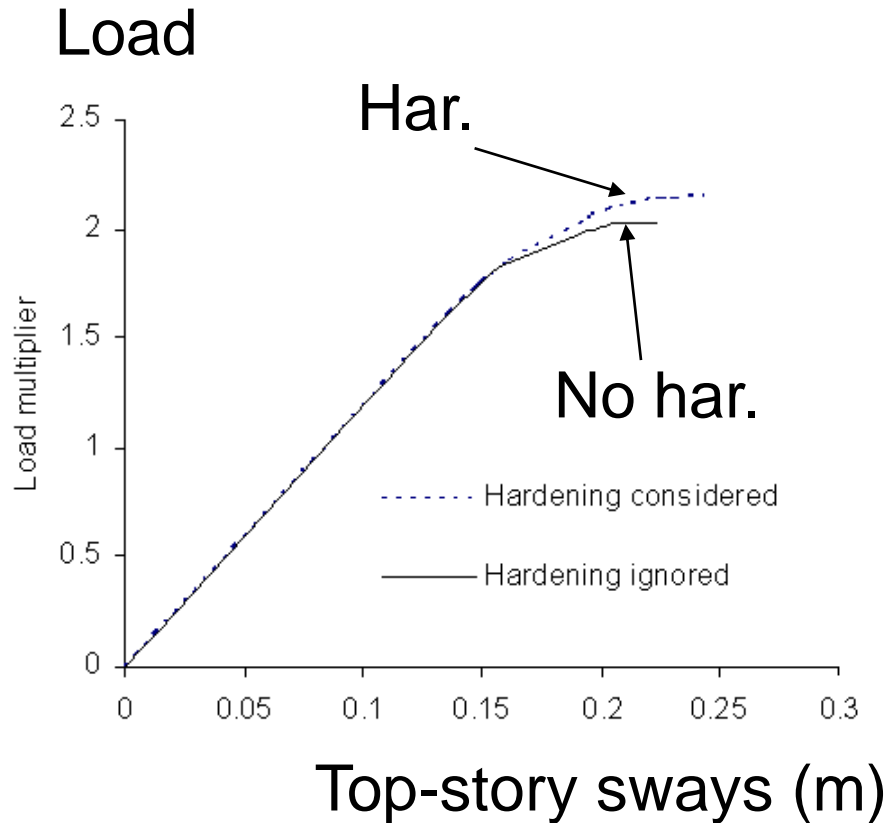
Instantaneous  
mechanism  
(limit analysis)

Incremental  
mechanism  
(shakedown  
analysis)

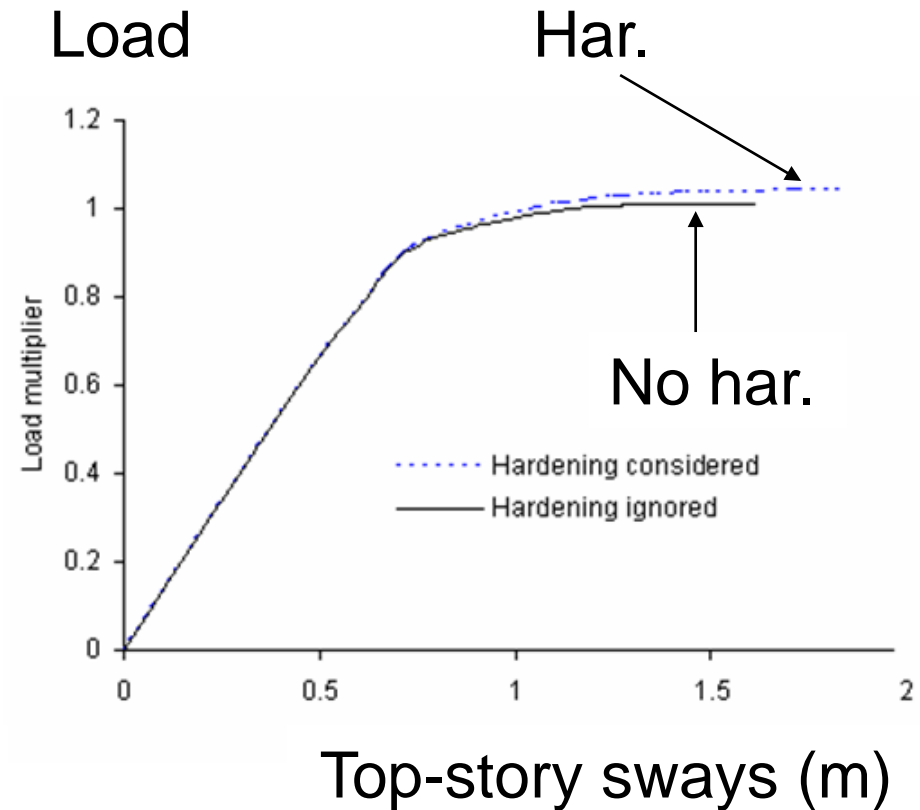
Altenating  
plasticity  
(shakedown  
analysis)

# Numerical examples

## Hardening effect



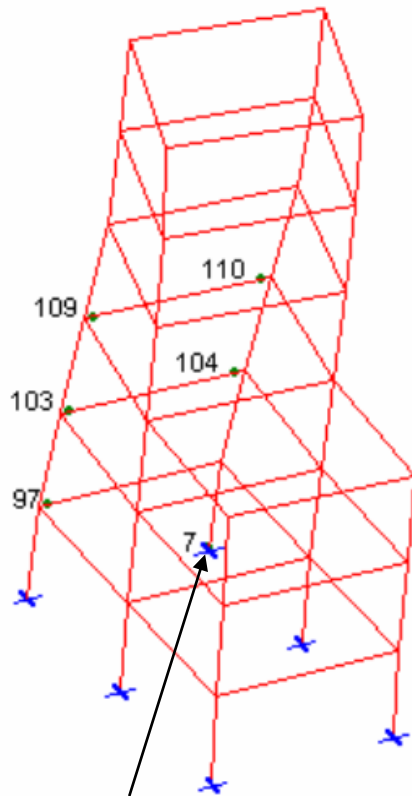
Frame I.1



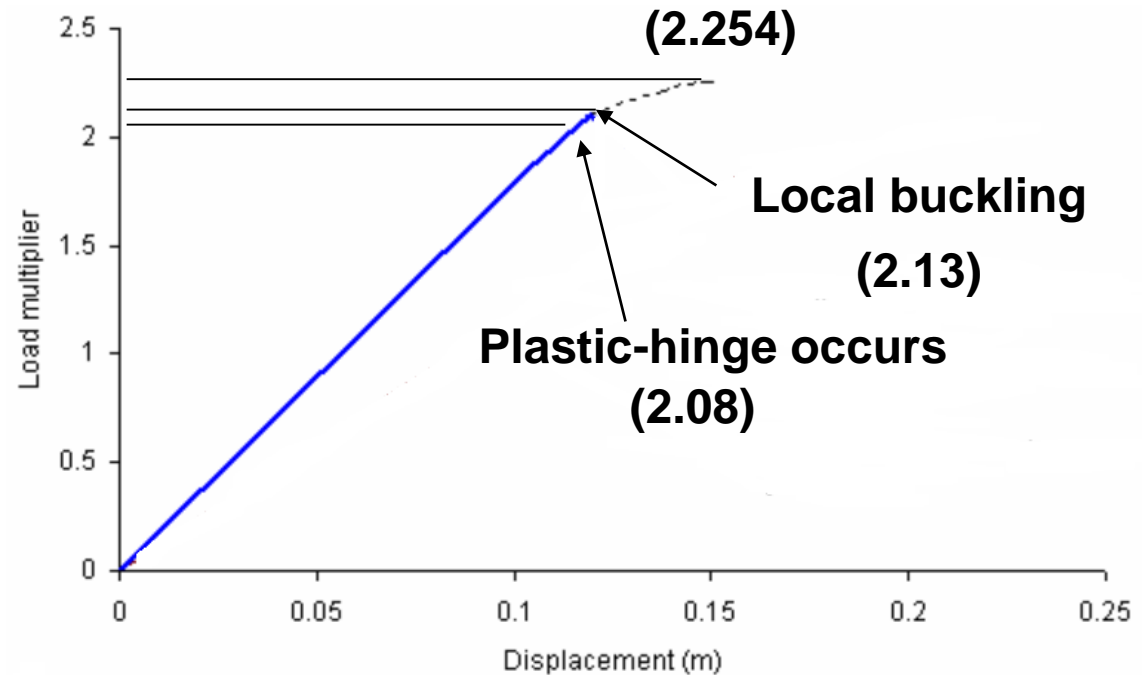
Frame I.2

# Numerical examples

## Local buckling check



**section 7**





# Numerical examples

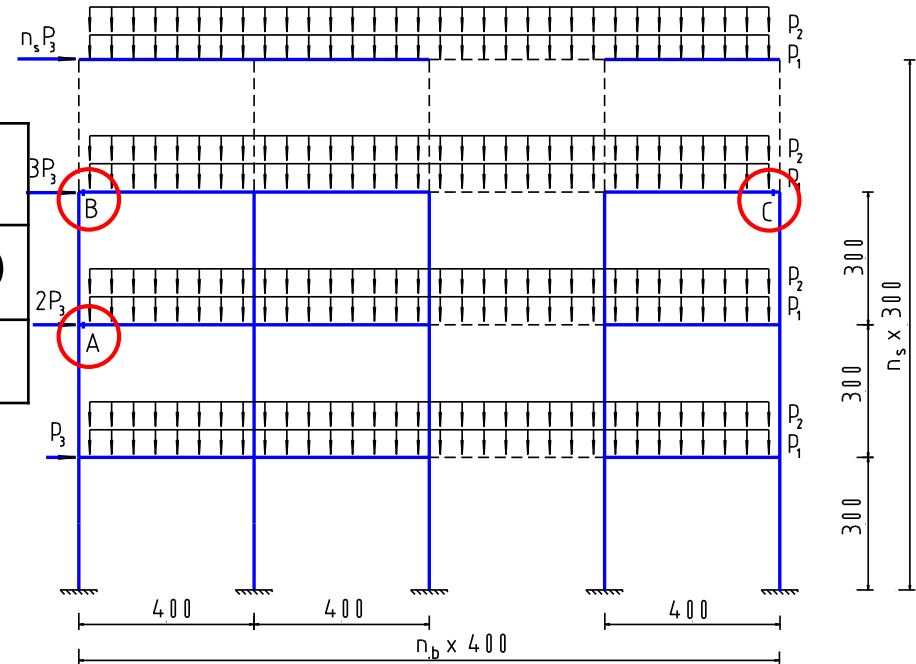
## II. 2-D bending frames analysis (Casciaro -2002)

### Mechanical properties

	E	I	$M_P$
Column	300000	540000	1800000
Beam	300000	67500	450000

### 4 frames:

- 1: 3 spans, 4 stories
- 2: 4 spans, 6 stories
- 3: 5 spans, 9 stories
- 4: 6 spans, 10 stories



### Load domain:

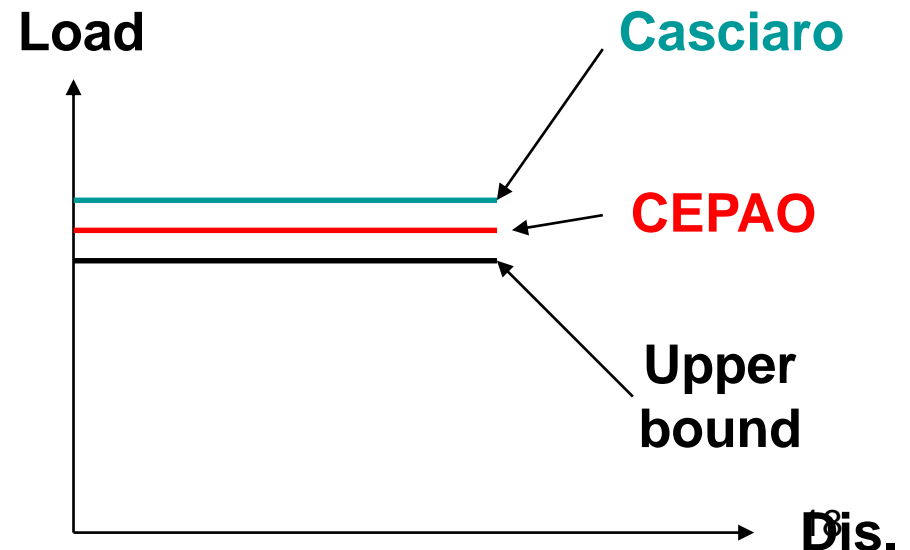
$$9\lambda \leq p1 \leq 10\lambda; 0\lambda \leq p2 \leq 5\lambda; -500\lambda \leq P3 \leq 500\lambda$$

# Numerical examples

Frame ( $n_s \times n_b$ )	Limit analysis			Shakedown analysis		
	Casciaro	CEPAO	Dif.	Casciaro	CEPAO	Dif.
3×4	2.4612	2.4612	0.0%	2.0134	2.0102	0.0%
4×6	1.8610	1.8610	0.0%	1.3993	1.2655	-10.5%
5×9	1.2000	1.2000	0.0%	0.7533	0.7076	-6.4%
6×10	1.1532	1.1532	0.0%	0.7209	0.6771	-6.5%

## Upper bounds

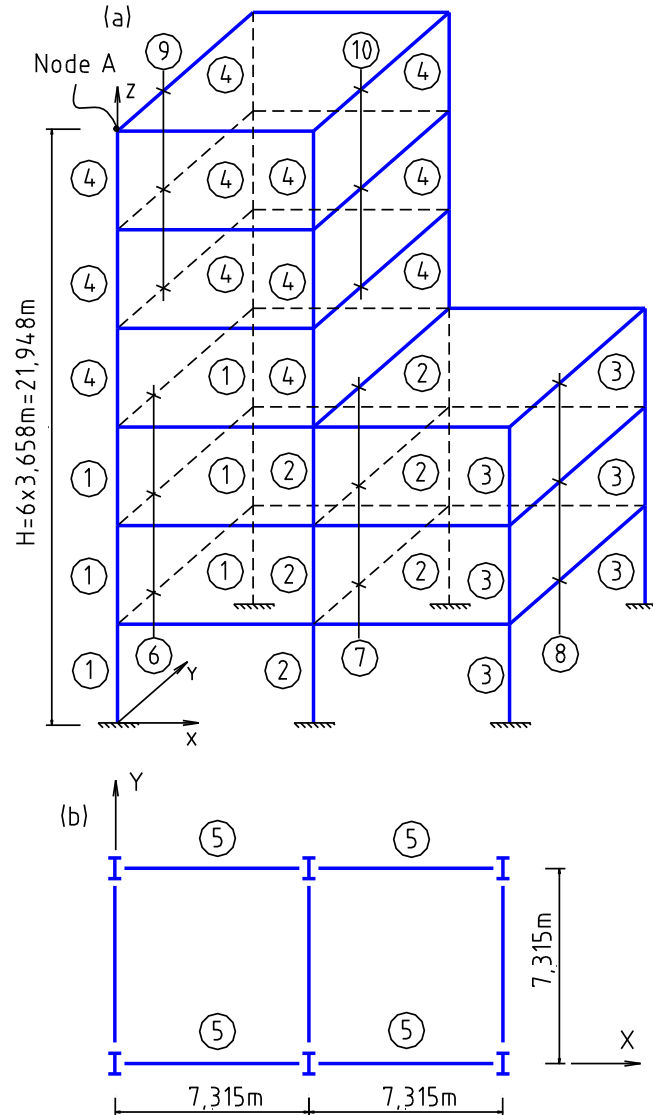
Alternating plasticity at	Load Multiplier
Section A (1)	1.1846
Section B (2)	0.6816
Section C (3)	0.6533



# Numerical examples

## III. 3-D rigid frames optimization

Frame III.1  
(shakedown design)



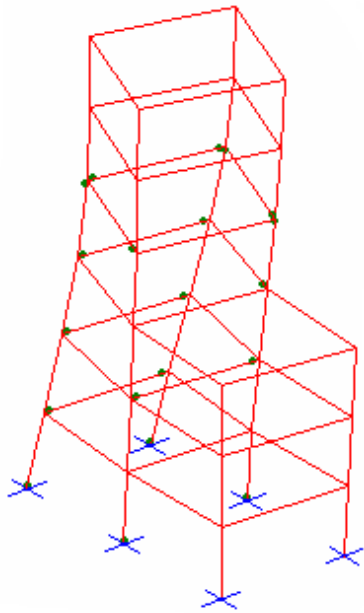
# Numerical examples

## Frame III.1(American sections)

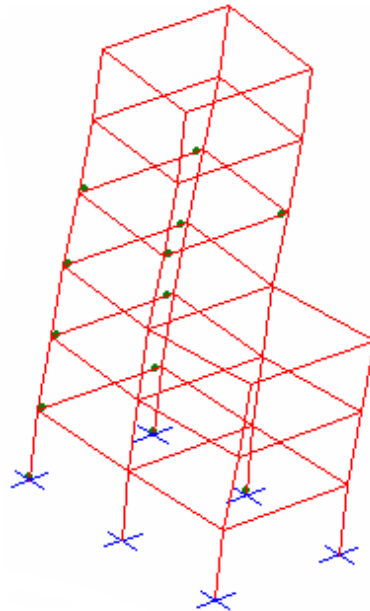
Design variable	Iteration					
	0 (initial)	1	2	3	4	5 (optimal)
1	W12X87	W16X89	W16X67	W16X67	W18X65	W21X62
2	W12X120	W21X101	W24X76	W24X76	W24X76	W24X76
3	W12X87	W24X55	W18X55	W21X44	W21X44	W21X44
4	W10X60	W18X60	W18X40	W16X40	W16X40	W16X40
5	W12X26	W12X21	W12X14	W12X14	W12X14	W12X14
6	W12X53	W18X60	W18X50	W18X50	W18X50	W18X50
7	W12X87	W14X61	W18X46	W21X44	W18X50	W18X50
8	W12X53	W14X48	W12X21	W12X19	W12X19	W12X19
9	W12X53	W14X43	W14X30	W16X26	W16X26	W16X26
10	W12X87	W18X45	W18X35	W16X31	W16X26	W16X26
Weight (kN)	295.2	244.1	178.0	170.8	170.4	169.4
Safely factor	1.670	2.283	1.785	1.784	1.767	1.754

# Numerical examples

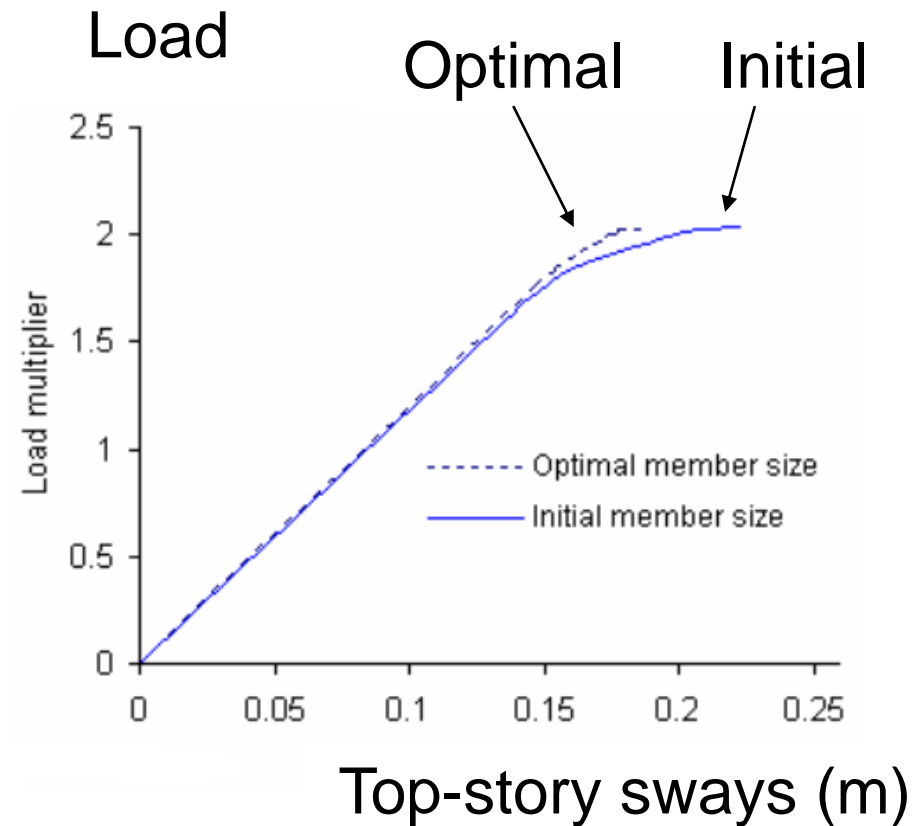
Optimal?



Initial  
member-size  
(295kN)



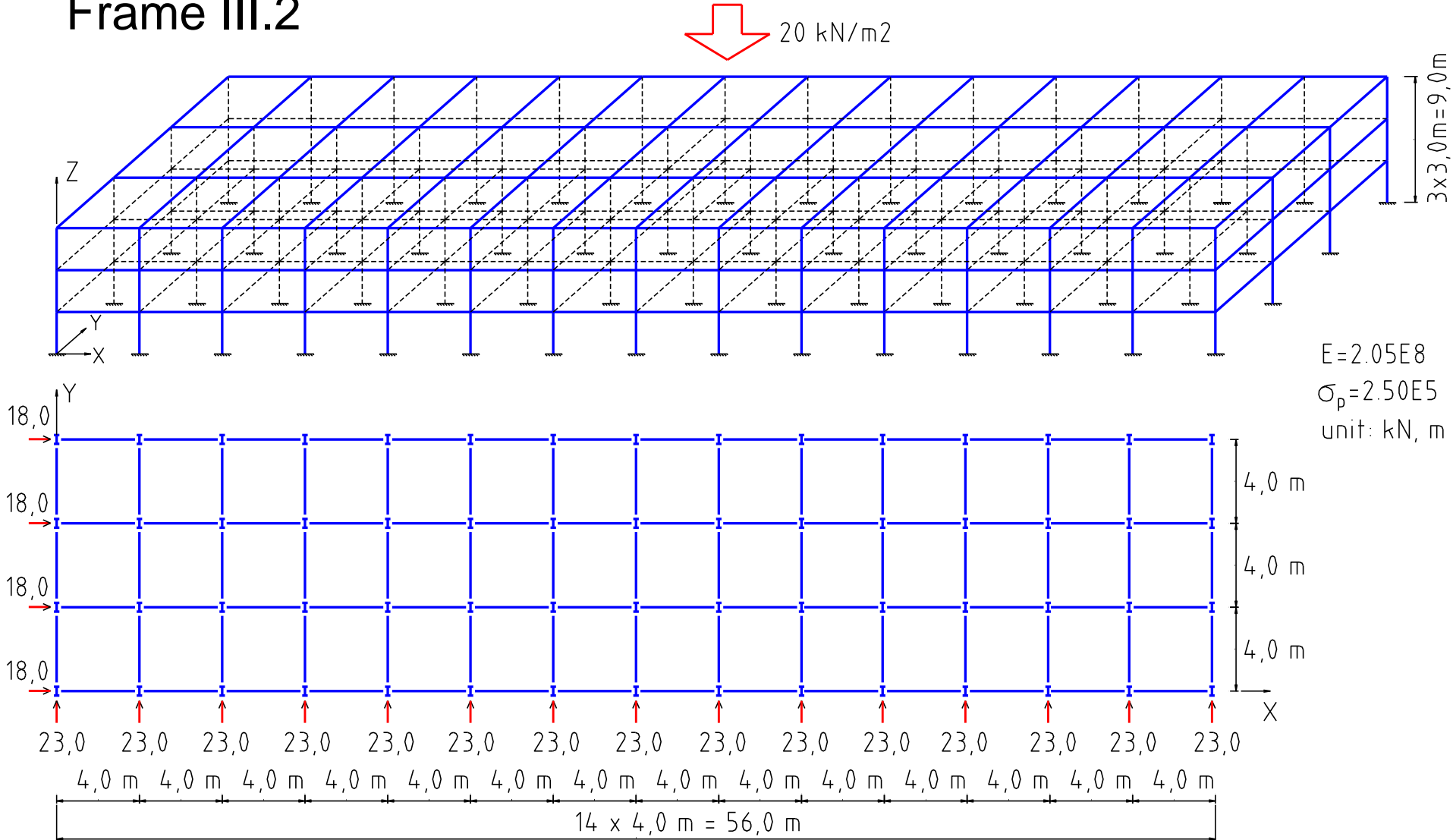
optimal  
member-size  
(169kN)



(Second-order analysis)

# Numerical examples

## Frame III.2



Columns of first story: ① Columns of 2nd story: ② Columns of third story: ③

Beams in Y direction: ④ Beams in X direction: ⑤

# Numerical examples

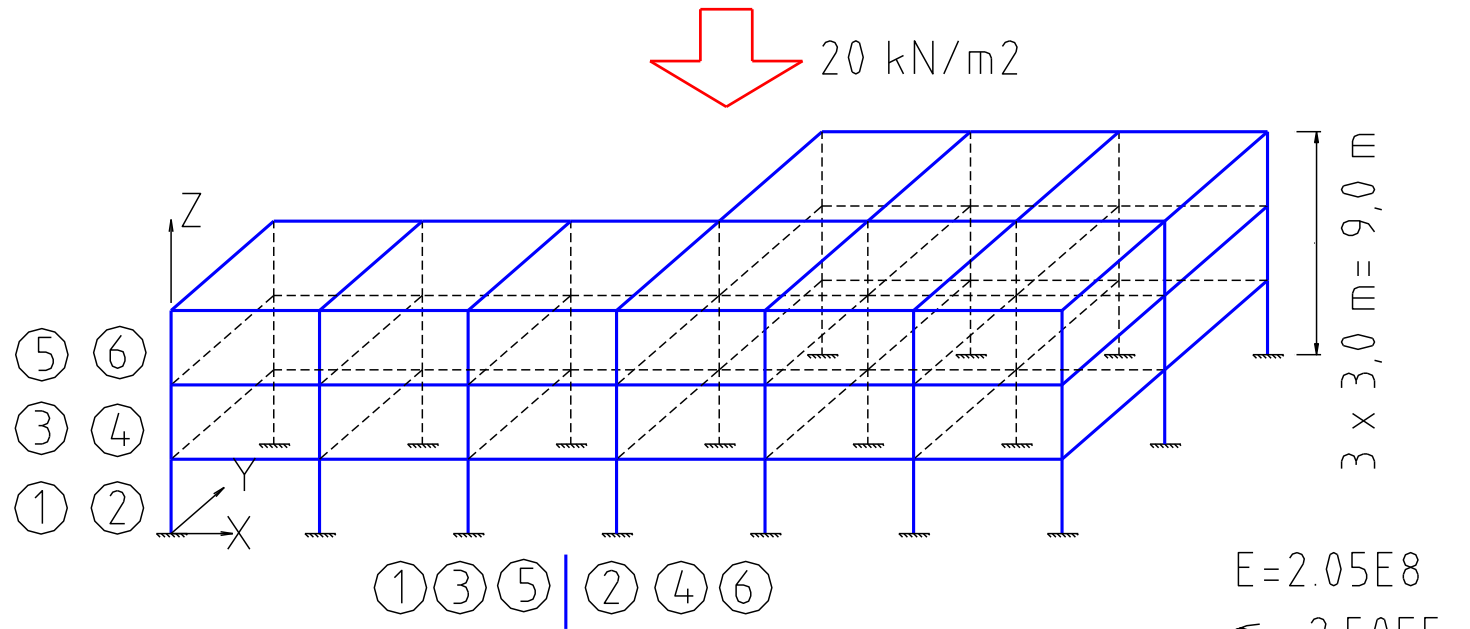
## Convergence!

Frame III.2 (American sections)

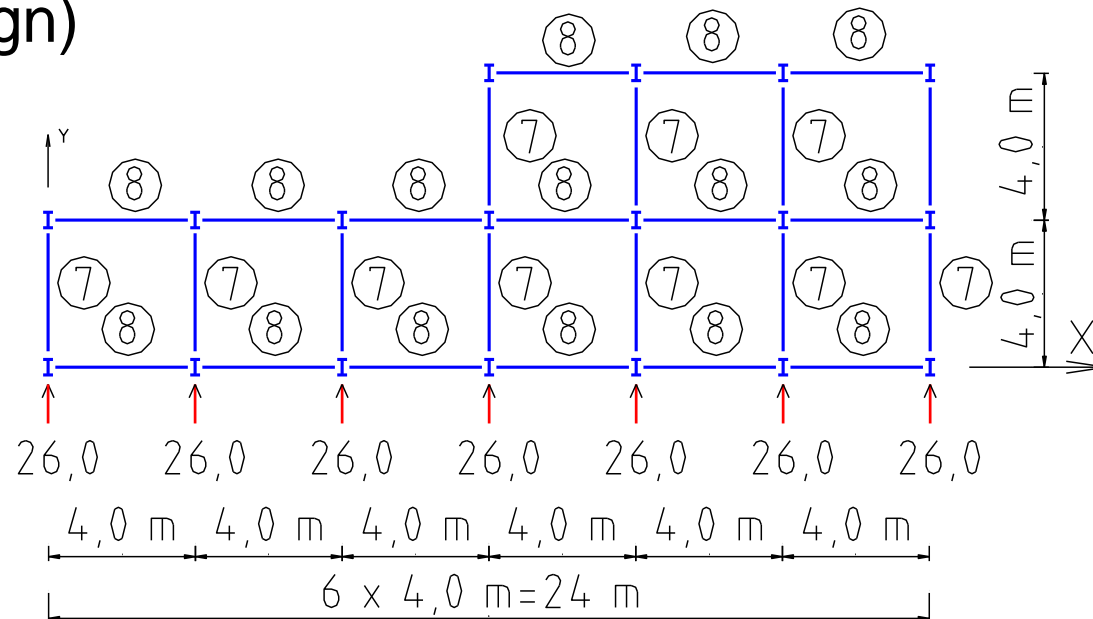
Design variable	Iteration		
	0 (initial)	1	2(optimal)
1	W6X9	W12X30	W12X26
2	W6X9	W8X21	W10X17
3	W6X9	W10X12	W10X12
4	W6X9	W6X9	W6X9
5	W6X9	W6X9	W6X9
Weight (kN)	238.06	333.83	312.69
Safely factor	0.42	1.52	1.32

**(Limit design)**

# Frame III.3



(Shakedown design)





# Numerical examples

## Frame III.3 (American sections)

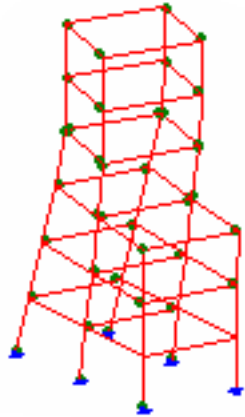
Design variable	Iteration						
	0 (initial)	1	2	3	4	5	6 (optimal)
1	W6X9	W14X43	W16X26	W16X26	W14X30	W14X30	W12X30
2	W6X9	W14X34	W14X22	W14X22	W14X22	W14X22	W14X22
3	W6X9	W14X38	W14X22	W14X22	W14X22	W14X22	W12X21
4	W6X9	W12X30	W12X19	W12X19	W12X19	W12X16	W12X14
5	W6X9	W16X36	W16X26	W16X26	W16X26	W16X31	W16X26
6	W6X9	W14X34	W12X21	W12X21	W12X21	W12X16	W12X14
7	W6X9	W12X30	W12X19	W12X14	W12X14	W12X14	W12X14
8	W6X9	W14X34	W12X19	W12X16	W12X14	W10X12	W10X12
Weight (kN)	64.40	243.03	178.0	123.83	119.94	111.40	107.77
Safely factor	0.26	1.87	1.14	1.09	1.17	1.17	1.15

## Frame III.3 (European sections)

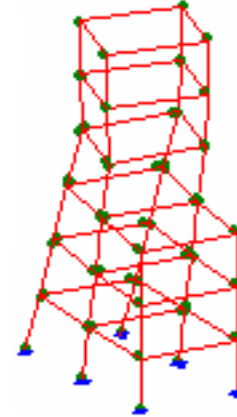
Design variable	Iteration					
	0 (initial)	1	2	3	4	5 (optimal)
1	IPE 80A	HE500AA	IPEA360	IPE330	IPE330	IPEA330
2	IPE 80A	IPEA500	IPEA330	IPEA300	IPE270	IPEA270
3	IPE 80A	IPE320A	IPE270	IPEA270	IPE240	IPE240
4	IPE 80A	IPN380	IPE270	IPEO220	IPEO200	IPEO200
5	IPE 80A	HP260X87	IPE240	IPEA240	IPEA240	IPEA240
6	IPE 80A	HE240B	IPEA240	IPEA220	IPEA200	IPEA200
7	IPE 80A	HP350X88	IPEA270	IPEA240	IPEA240	IPEA240
8	IPE 80A	HE320AA	IPEA240	IPEA240	IPEA200	IPEA180
Weight (kN)	23.73	391.07	146.57	125.29	115.44	107.00
Safely factor	0.056	3.26	1.39	1.24	1.19	1.03

# Numerical examples

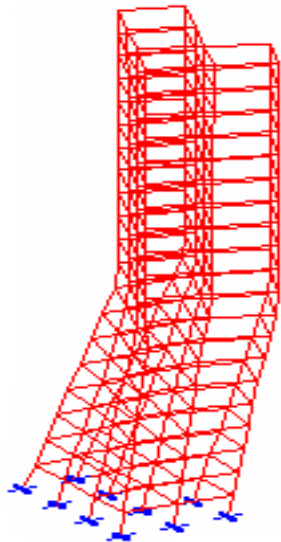
## IV. Convergence of SBS and Direct methods



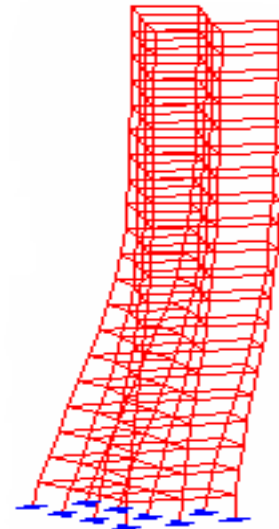
Direct method (**2.412**)



Step-by-step method (**2.489**)

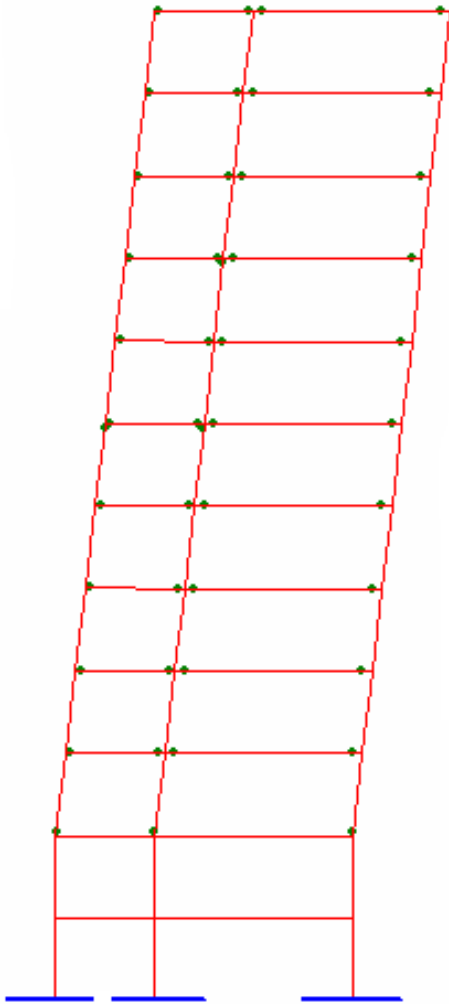


Direct method (**1.698**)

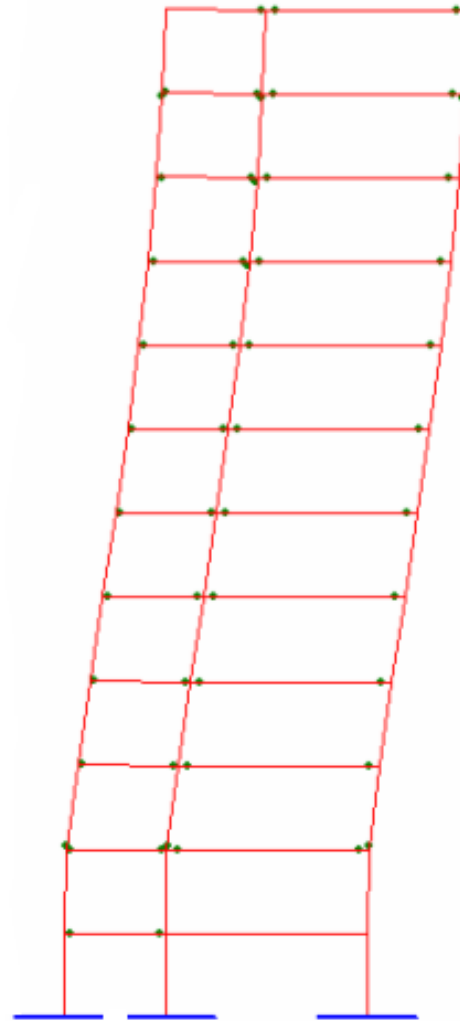


Step-by-step method (**1.689**)

# Numerical examples

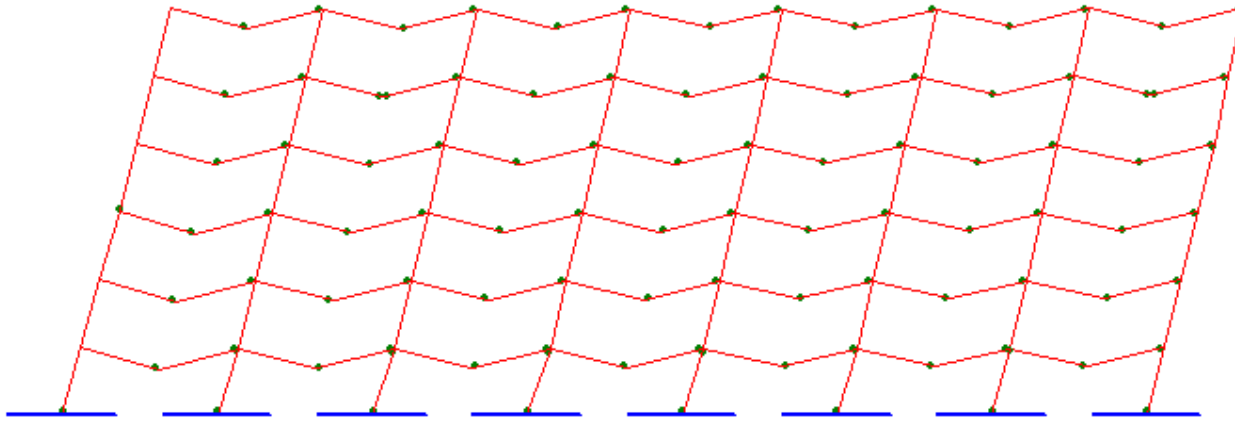


Direct method (2.126)

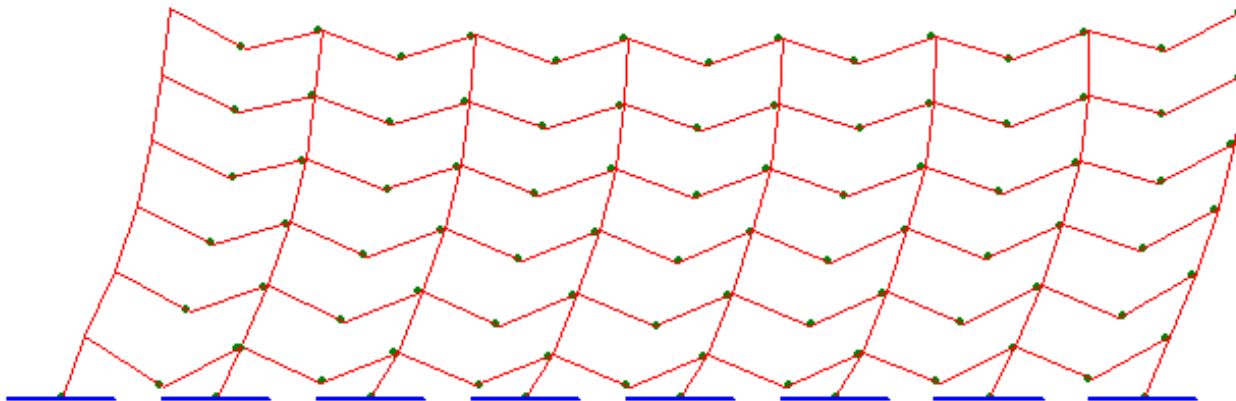


Step-by-step method (2.175)

# Numerical examples

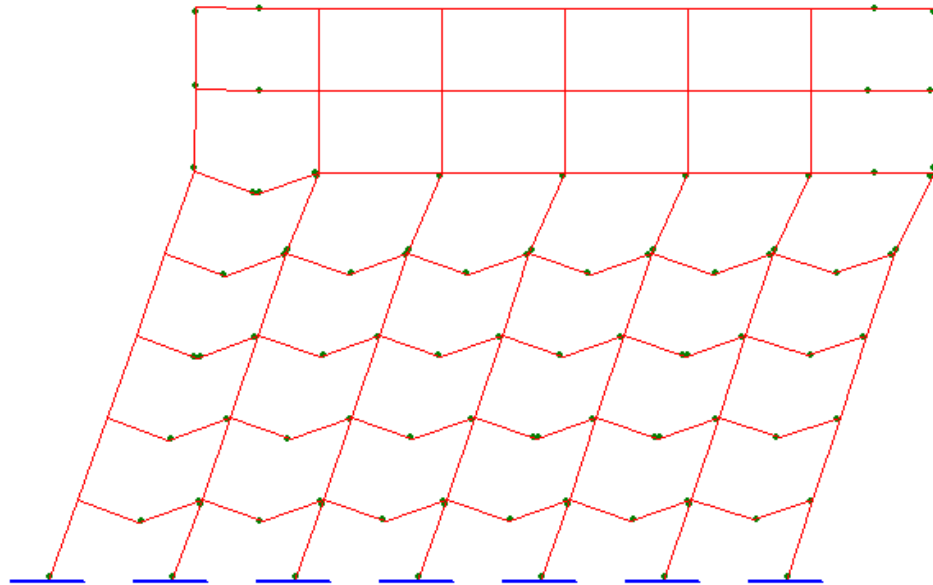


Direct method (2.469)

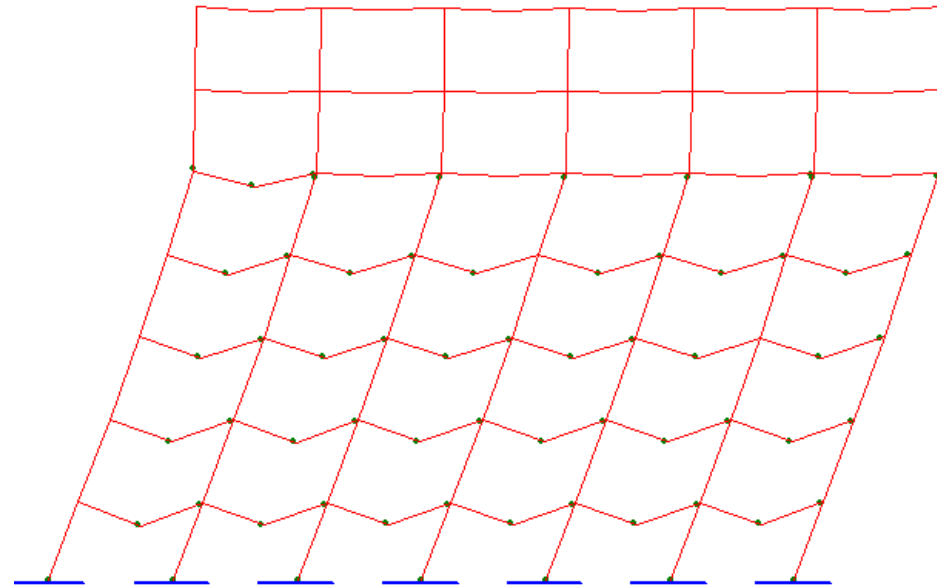


Step-by-step method (2.402)

# Numerical examples



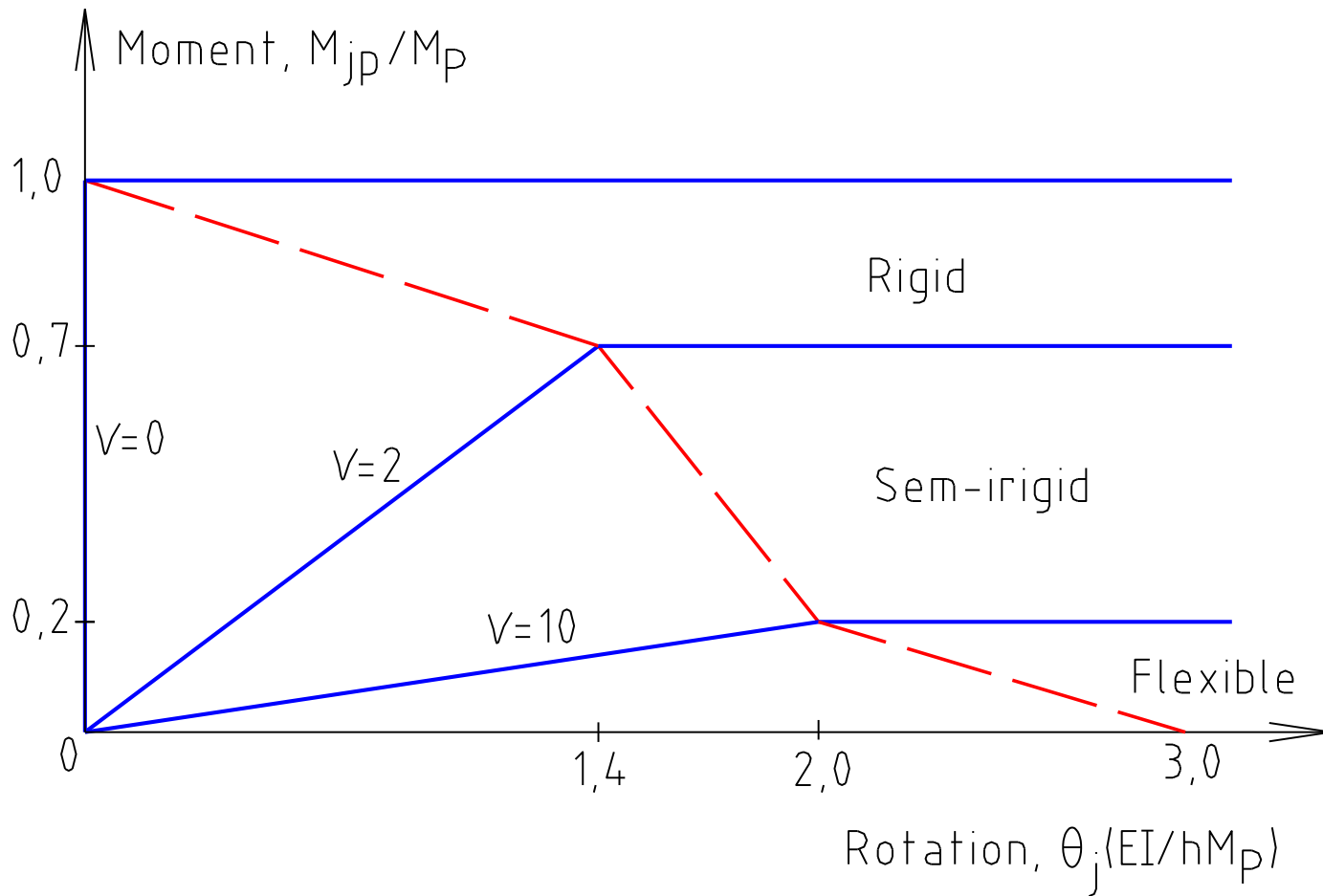
Direct method (2.226)



Step-by-step method (2.264)

# Numerical examples

## V. 2-D semi-rigid frames



Moment-rotation relationship for connexions  
(Bjorhovde-1990)

# Numerical examples

## V. 2-D semi-rigid frames, Tin Loi 1993

### Limit and shakedown analysis

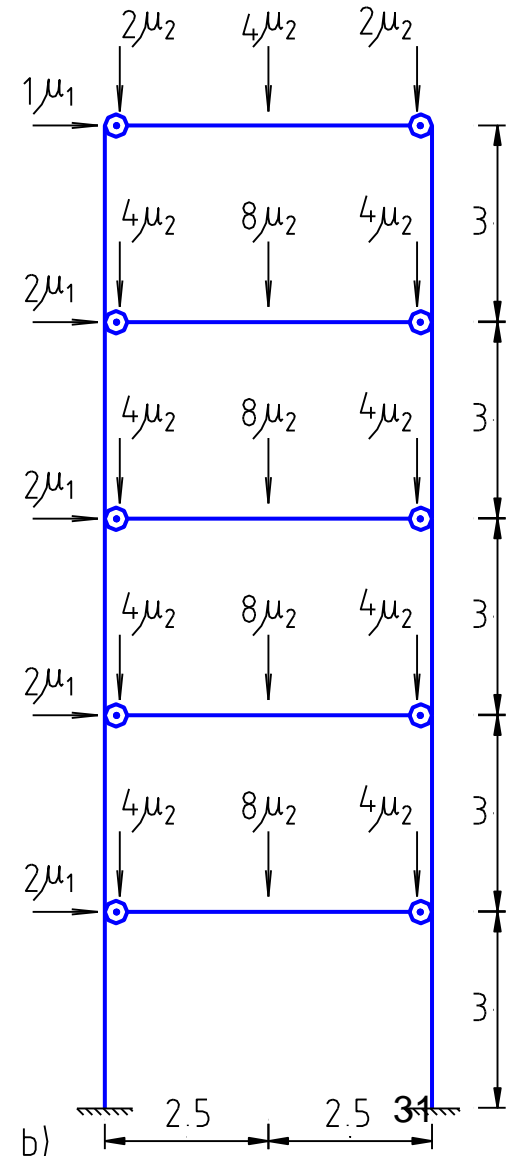
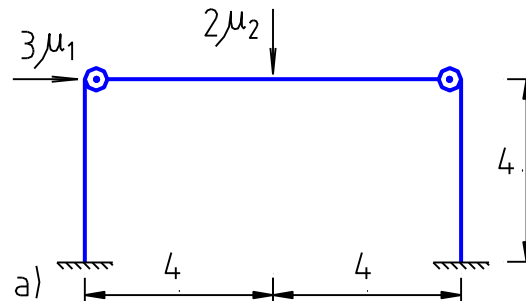
*Frame V.1a:*  $E = 2.1E7$ ;  $I = 118.5E-6$ ;  
 $M_p = 20$ ;  $h = 0.3$ ;

*Frame V.1b:* column:  $E = 2.1E7$ ;  $I = 85.2E-6$ ;  $M_p = 10$ ;  
 beam:  $E = 2.1E7$ ;  $I = 118.5E-6$ ;  $M_p = 20$ ;  $h = 0.3$ .

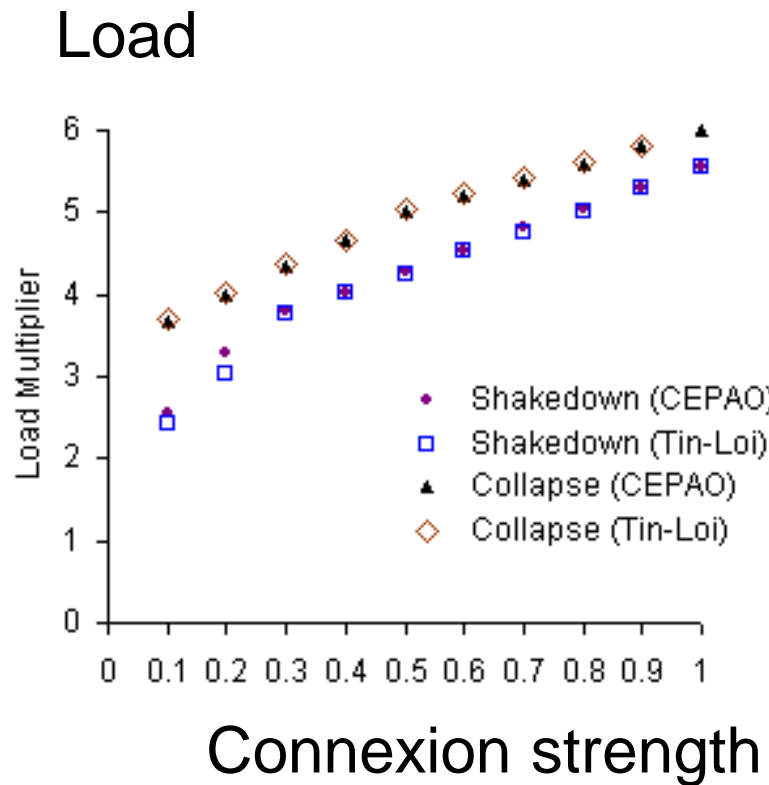
Load domain:

$$0 \leq \mu_1 \leq \mu$$

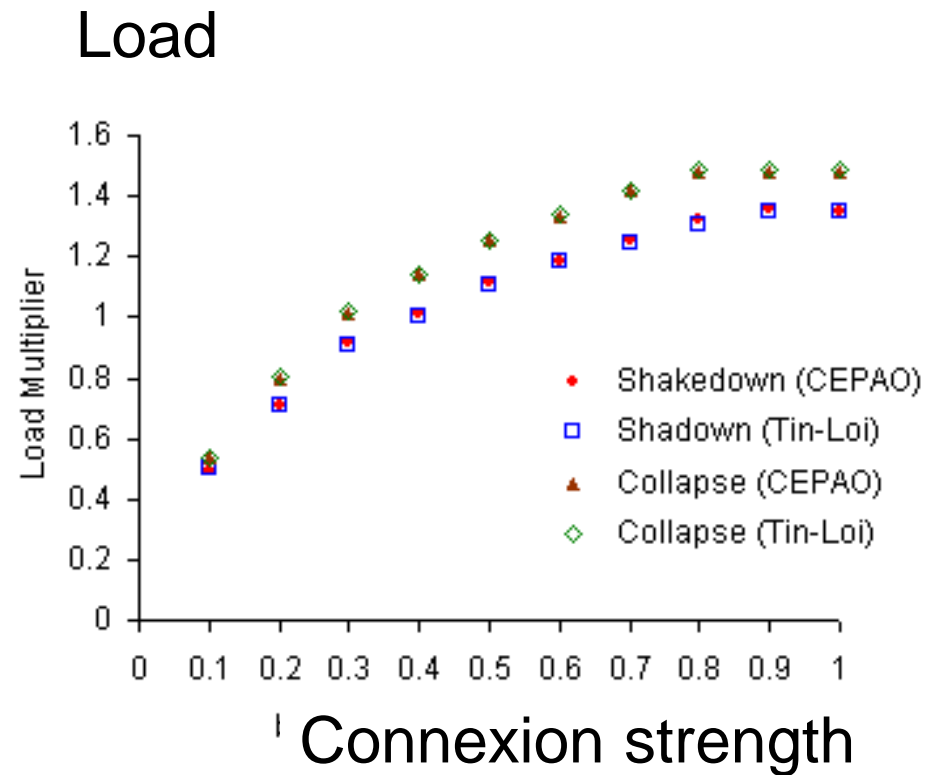
$$0 \leq \mu_2 \leq \mu$$



# Numerical examples



Frame V.1a



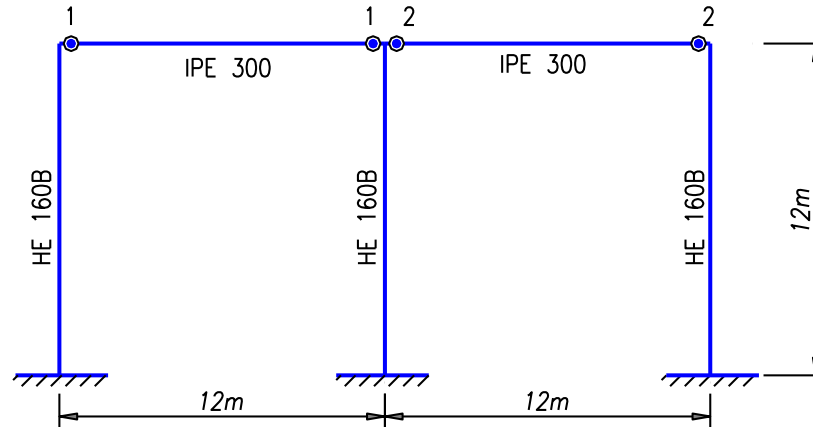
Frame V.1b



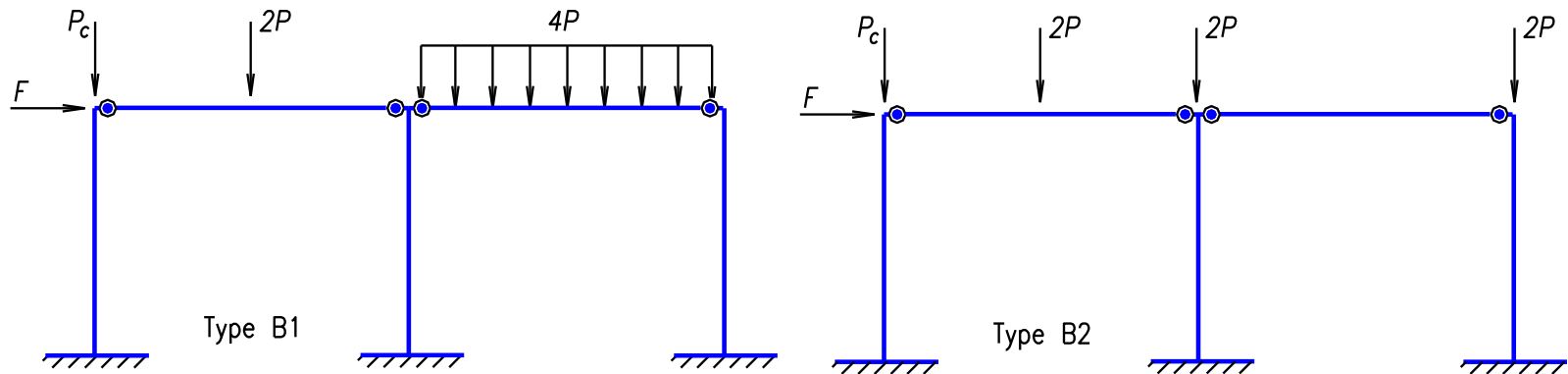
# Numerical examples

## V. 2-D semi-rigid frames, Jaspart – 1991 (using FINELG)

Type 1



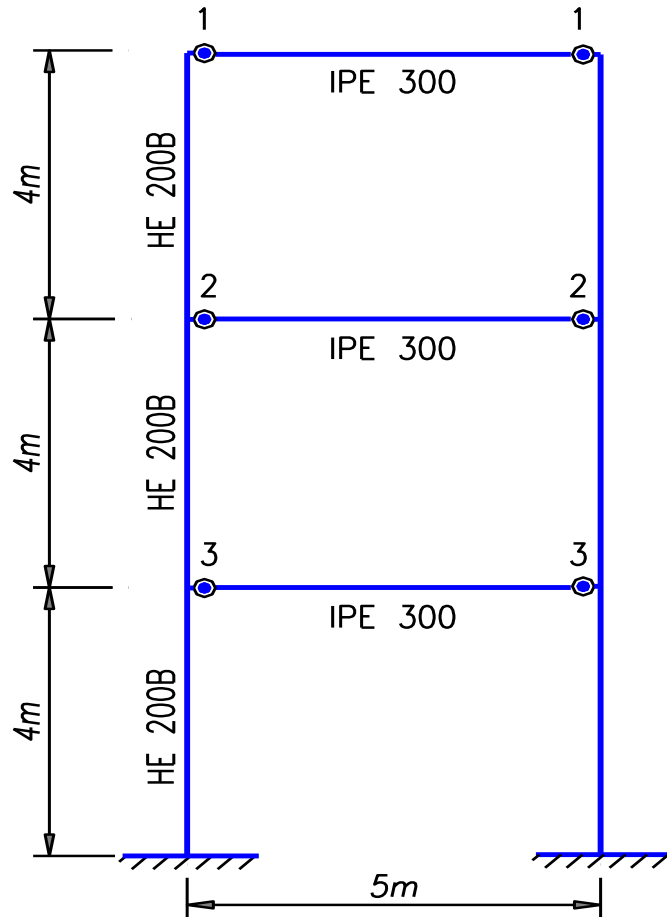
Geometry



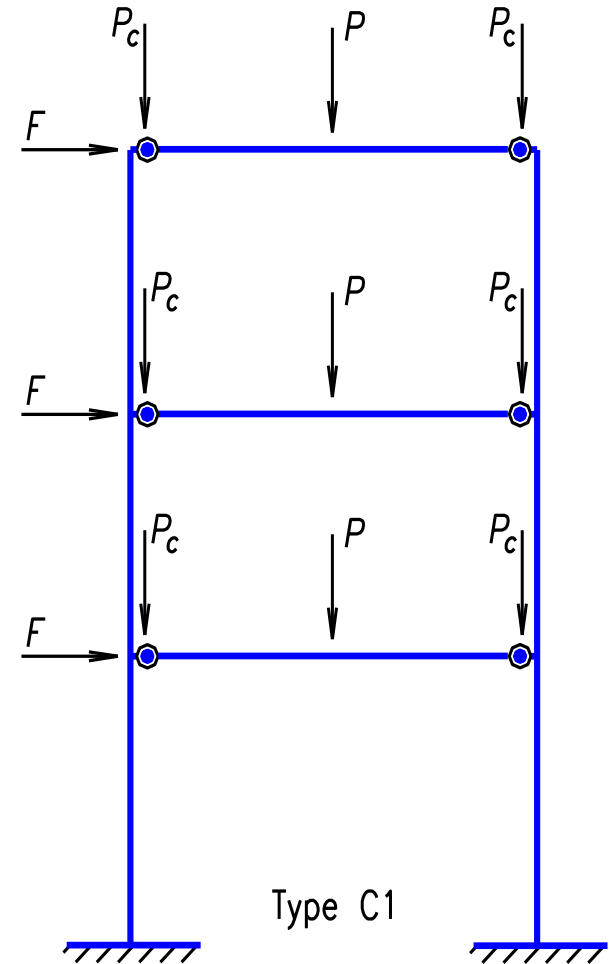
Loading

# Numerical examples

## Type 2



Geometry

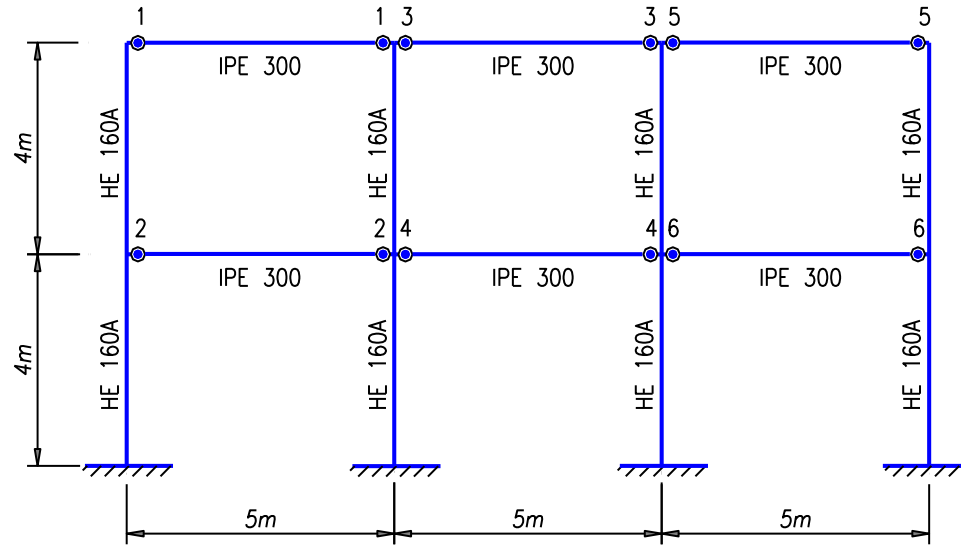


Loading

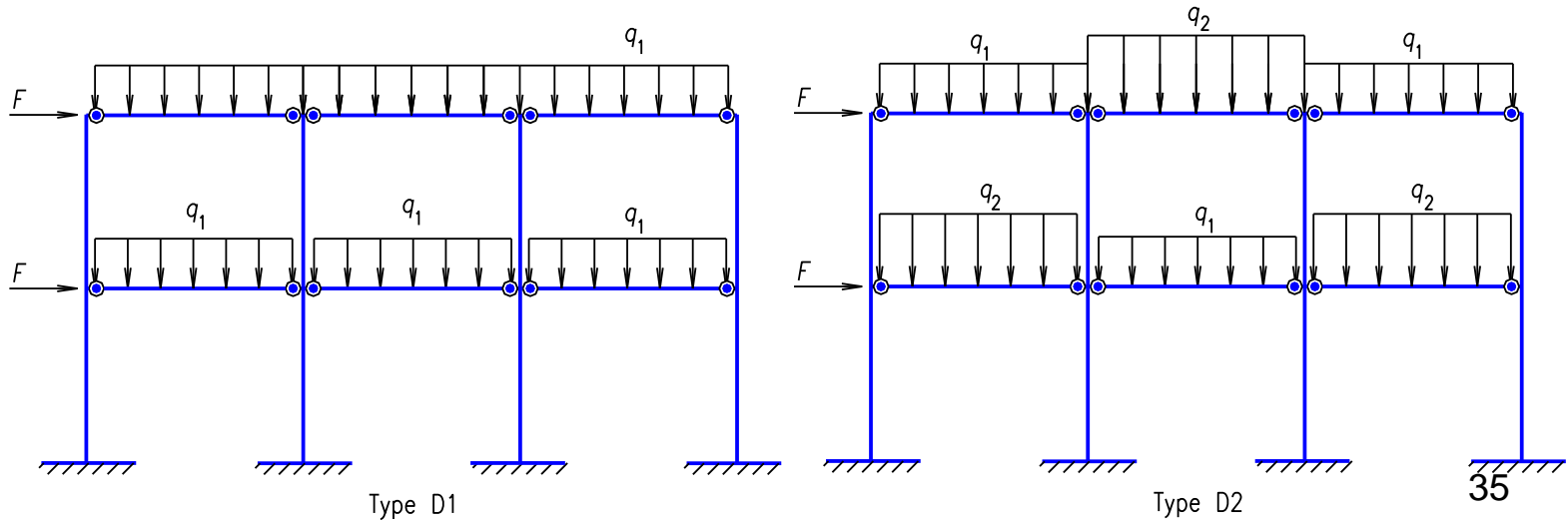
# Numerical examples

## Type 3

Geometry



Loading



# Numerical examples

## Type 1

Frame	Loading				Assemblages		Load multipliers		
	Type	F(kN)	P(kN)	$P_c$ (kN)	$M_{i,p1}$	$M_{i,p2}$	FINELG	CEPAO	Direct
								SBS	
BP1	1	12.0	12.0	36.0	147.7	105.0	2.98	2.96	2.91
BP2	1	12.0	12.0	36.0	147.7	77.0	2.96	2.93	2.90
BP3	1	12.0	12.0	36.0	147.7	39.0	2.59	2.59	2.58
BP4	1	12.0	12.0	36.0	105.0	77.0	2.79	2.78	2.75
BP5	1	12.0	12.0	36.0	77.0	105.0	2.69	2.67	2.64
BP6	1	12.0	12.0	36.0	105.0	147.7	2.82	2.78	2.77
BP7	1	12.0	12.0	36.0	0.0	147.7	2.05	2.17	2.06
BP8	1	12.0	12.0	36.0	39.0	147.7	2.51	2.50	2.47
BP9	1	12.0	12.0	36.0	105.0	39.0	2.59	2.59	2.58
BP10	1	12.0	12.0	36.0	77.0	77.0	2.66	2.65	2.62
BP11	2	12.0	12.0	36.0	147.7	39.0	2.74	2.72	2.67
BP12	2	12.0	12.0	36.0	147.7	77.0	2.96	2.94	2.90
BP13	2	12.0	12.0	36.0	147.7	105.0	2.98	2.96	2.91
BP14	1	12.0	6.0	18.0	77.0	147.7	3.43	3.41	3.35

# Numerical examples

## Type 2

Frame	Loading				Assemblage	Load multipliers			
	Type	F(kN)	P(kN)	P <sub>c</sub> (kN)		M <sub>i,p1</sub> =M <sub>i,p2</sub> =M <sub>i,p3</sub>	FINELG	CEPAO	
								SBS	Direct
CP1	1	10.0	100.0	50.0	25.0	1.35	1.32	1.31	
CP 2	1	10.0	100.0	50.0	50.0	1.50	1.47	1.46	
CP 3	1	10.0	100.0	50.0	75.0	1.65	1.61	1.60	
CP 4	1	10.0	100.0	50.0	100.0	1.80	1.75	1.74	
CP 5	1	10.0	100.0	50.0	125.0	1.96	1.90	1.88	
CP 6	1	10.0	100.0	50.0	147.58	2.09	2.03	2.00	
CP 7	1	2.0	100.0	50.0	100.0	1.98	1.98	1.97	
CP 8	1	20.0	40.0	80.0	100.0	1.88	1.77	1.75	

# Numerical examples

## Type 3

Frame	Loading				Assemblage		Load multipliers	
	Type	F(kN)	q <sub>1</sub> (kN/m)	q <sub>2</sub> (kN/m)	M <sub>i,p1</sub> =...=M <sub>i,p6</sub> (kNm)	FINELG	CEPAO	
							SBS	Direct
DP1	1	5.0	40.0	-	25.0	1.38	1.38	1.38
DP2	1	5.0	40.0	-	37.5	1.48	1.48	1.47
DP3	1	5.0	40.0	-	51.6	1.59	1.59	1.58
DP4	1	5.0	40.0	-	120.0	1.89	1.87	1.85
DP5	1	25.0	20.0	-	120.0	2.30	2.04	1.99
DP6	2	20.0	20.0	40.0	120.0	2.14	1.78	1.75

# Numerical examples

## V. 2-D semi-rigid frames

### Load domain

- **Shakedown:**

a)  $0 \leq \lambda_1 \leq 1, 0 \leq \lambda_2 \leq 1$

b)  $-1 \leq \lambda_1 \leq 1, 0 \leq \lambda_2 \leq 1$ .

- **Limit:**  $\lambda_1 = \lambda_2 = \lambda$ ;

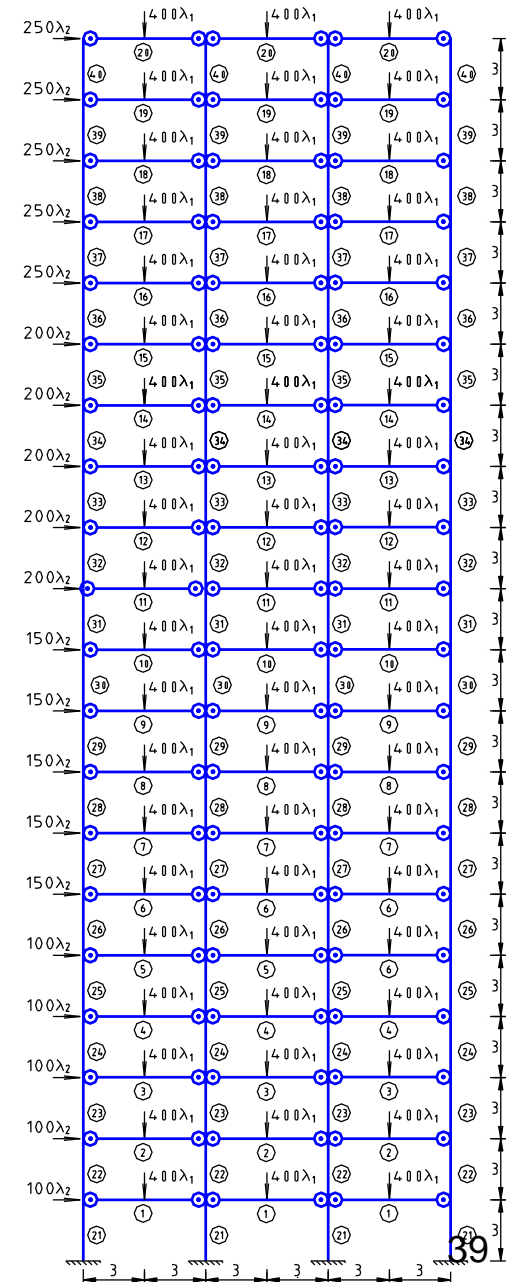
### Groups of elements:

- **Optimal:** 40 different groups of elements, load factor  $\lambda = 0.25$

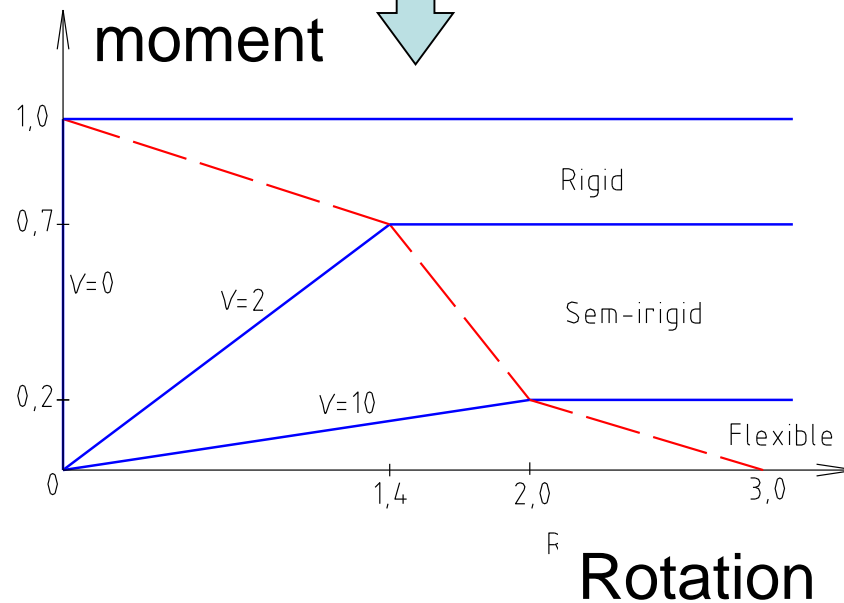
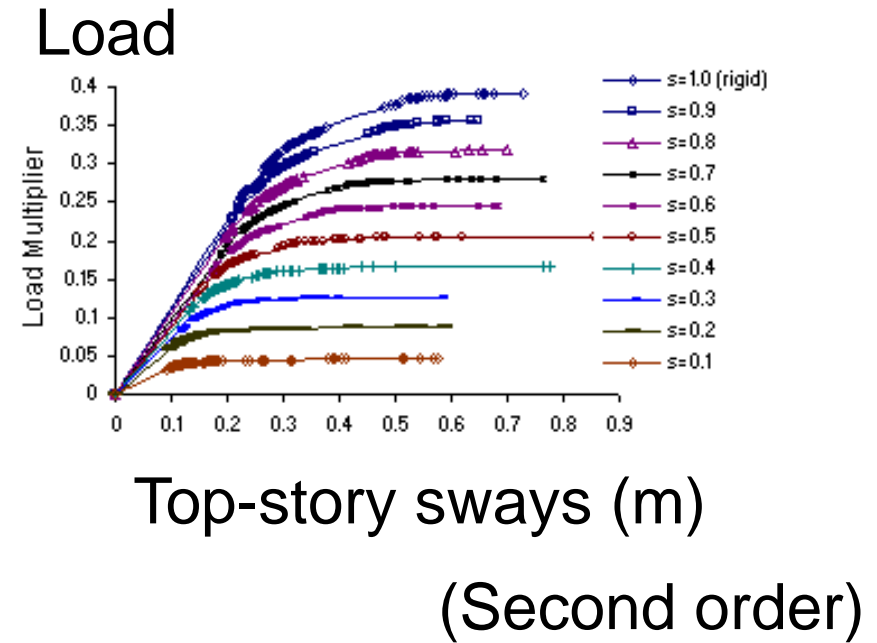
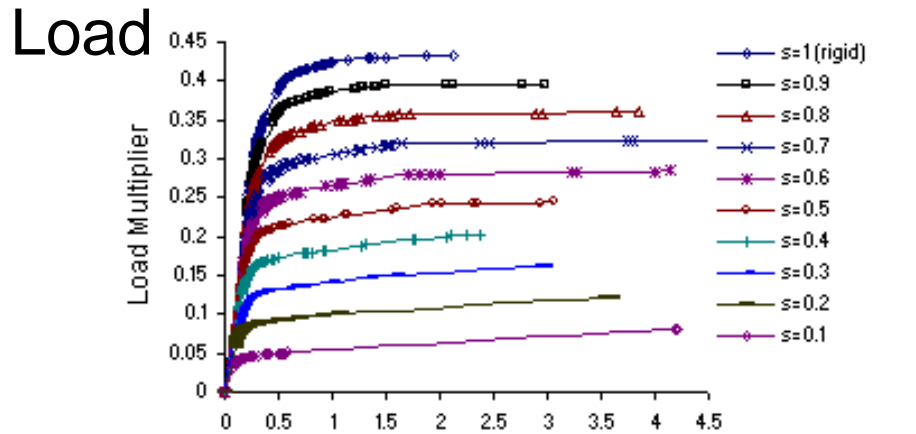
- **Analysis:** 8 different groups of elements:

Groups	1-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40
Profile	IPE550	IPE500	IPE450	IPE330	HE600A	HE550A	HE450A	HE360

$E=2E8, \sigma_p=2E5$



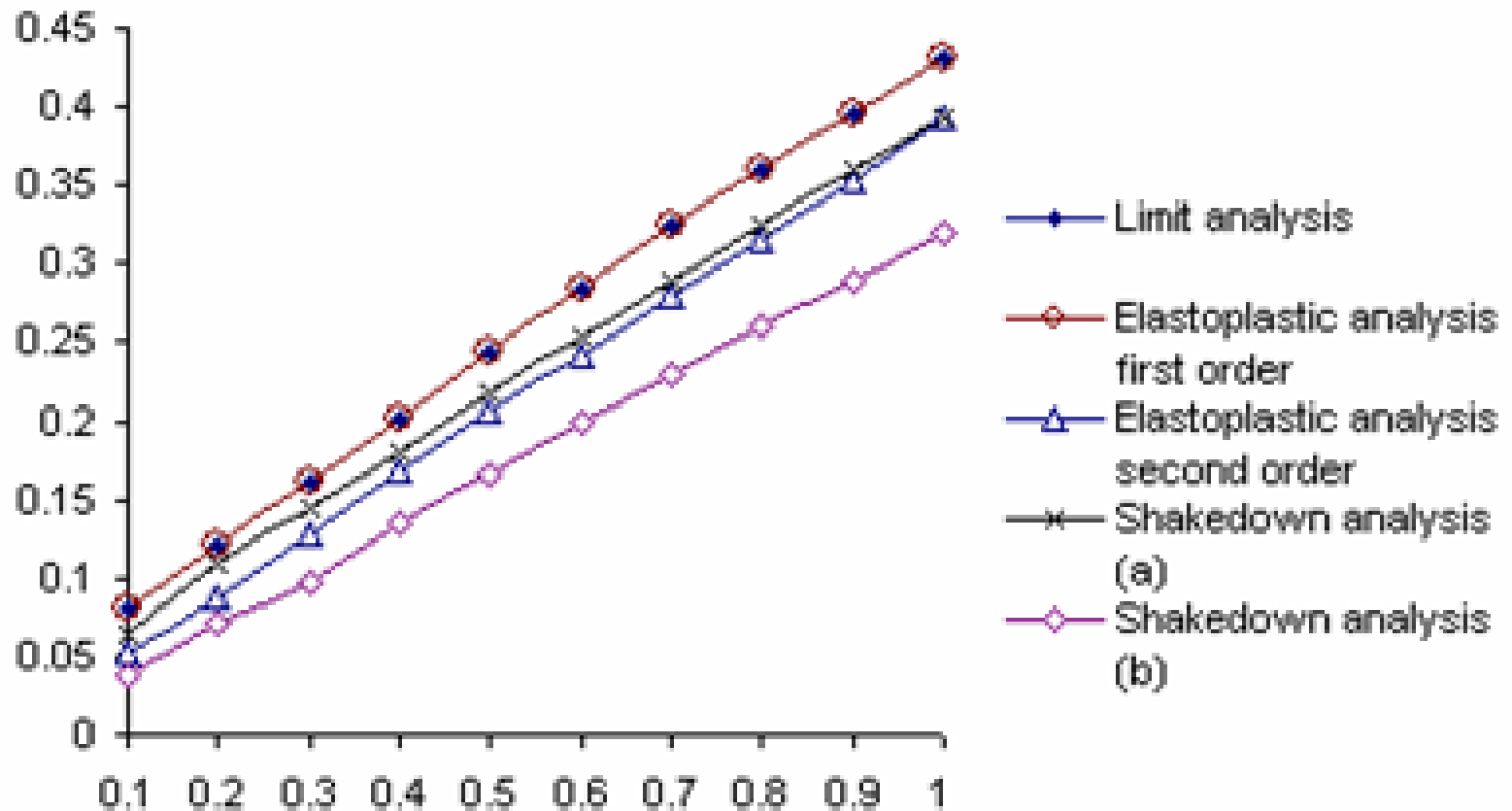
# Numerical examples





- Numerical examples

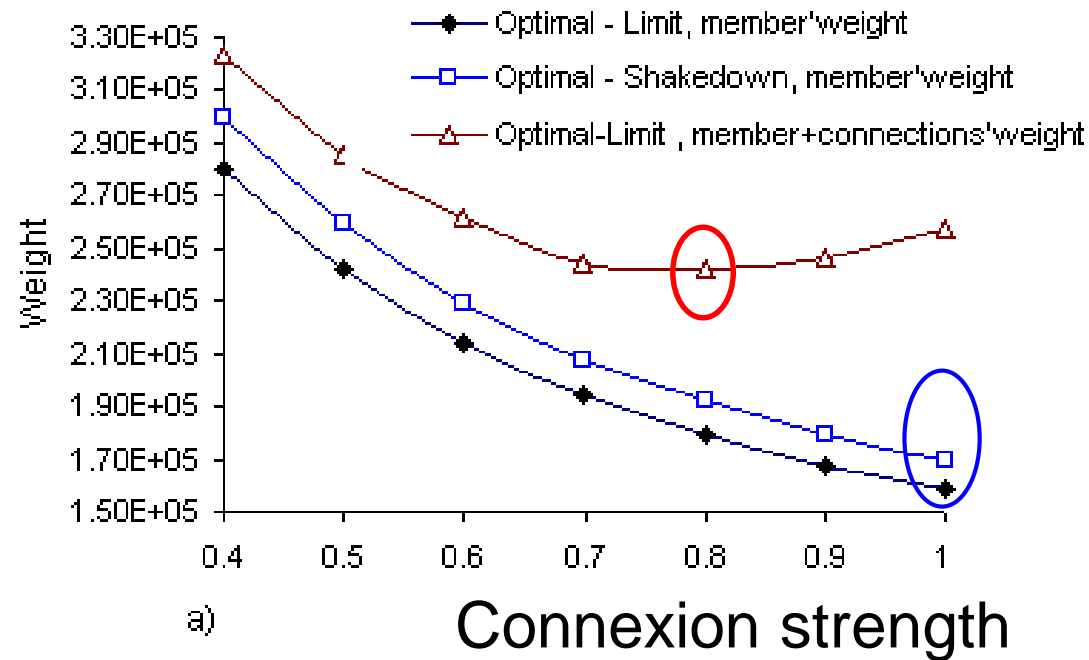
Load factors



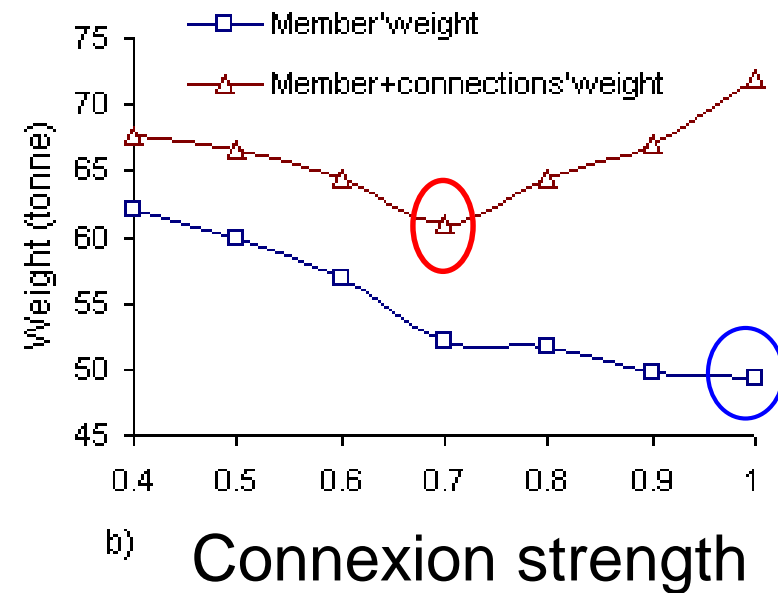
Joint strength

# Numerical examples

## Theoretic weight



## Real weight



Variation of weight according to connexion strengths

# Concluding remarks

**Thank you for your attention!**